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(Working Papers)

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Dedicated to

PROF. H. R. KRISHNA IYENGAR

M E S COLLEGE FACULTY & STUDENTS EDITION

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Classification of Second Order Partial Differential Equations of more than two variables

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Abstract: Classification of second order partial differential equations is based on geometric structures called as conic sections and associated quadratic equation. This classification for two variables is explained very well in all text books whereas more than two variables is difficult. Classification of PDE of more than two variables is studied and presented with example.

Keywords: Conic sections, Parabolic, Hyperbolic, Elliptic, Semi-definite, Definite, Infinite quadratic forms.

1 Introduction

A conic is the locus of a point in a plane such that the ratio of its distance from a fixed point in the plane to its distance from a fixed line in the plane is constant. When a right circular cone is cut by planes in different positions, the sections obtained are the curves - Circle, Parabola, Ellipse and Hyperbola. A circle is the locus of a point which moves in a plane such that its distance from a fixed point in the plane is constant. The fixed point is called the centre and the constant distance is called the radius of the circle.

$$x^2 + y^2 = a^2$$

is the equation of the circle whose centre is the origin and radius is a. As Parabola, Ellipse, Hyperbola are sections of a cone by a plane, these curves are known as conic sections or simply conics. [2, 3]

(a) Parabola: $y^2 = 4ax$.

(b) Ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

(c) Hyperbola: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

2 Classification of Partial Differential Equations

The general form of second order linear PDE in two variables is given by [6, 7]

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g, \tag{1}$$

where a, b, c, d, e, f, g are functions of x and y only. If $g = 0$, then (1) is homogeneous. If $g \neq 0$, then (1) is non-homogeneous. Consider the operator $L(u)$ called as principal part of (1) consisting of second order terms (higher order terms) i.e,

$$L(u) = au_{xx} + 2bu_{xy} + cu_{yy}. \tag{2}$$

We can also write (2) as

$$L = aD_x^2 + 2bD_xD_y + cD_y^2 \quad \text{where} \quad D_x = \frac{\partial}{\partial x}, \tag{3}$$

with associated Quadratic form as

$$Q(\xi, \eta) = a\xi^2 + 2b\xi\eta + c\eta^2 = \eta^2 q(\varsigma), \tag{4}$$

$$\text{where} \quad q(\varsigma) = a\varsigma^2 - 2b\varsigma + c, \quad \varsigma = \frac{\xi}{\eta}. \tag{5}$$

The Quadratic form $Q(\xi, \eta)$ is called the characteristic form of (1) which are of three different types namely Definite, Semidefinite and Indefinite as given below.

(i) **Definite**: Q vanishes only at $\xi = \eta = 0$

$$Q = \begin{cases} +ve & \forall\, \xi, \eta \quad \text{(positive definite)} \\ -ve & \forall\, \xi, \eta \quad \text{(negative definite)} \end{cases}$$

If the equation $q(\varsigma) = 0$ has no real roots then (4) is called Elliptic i.e., $b^2 - 4ac < 0$ at (x, y).

(ii) **Semidefinite**: Q preserves the same sign but vanishes for values of ξ, η other than $\xi = \eta = 0$. If the equation $q(\varsigma) = 0$ has one double root then (4) is called Parabolic i.e, $b^2 - 4ac = 0$ at (x, y).

(iii) **Indefinite**: Q takes both positive and negative values. If the equation $q(\varsigma) = 0$ has two distinct real roots then (4) is called Hyperbola i.e., $b^2 - 4ac = 0$ at (x, y).

Depending on the roots of characteristic equations one can reduce the given equations into canonical forms as follows.

1. **Hyperbolic Equation**: Consider the characteristic equation

$$q(\varsigma) = 0. \tag{6}$$

If it has two distinct real roots say $\varsigma_1(x, y)$ and $\varsigma_2(x, y)$, then this gives two families of characteristic curves in the xy plane with characteristic direction $\varsigma_i(x, y)$ and is defined by the equation

$$\frac{dy_i}{dx} = \varsigma_i(x, y); \quad i = 1, 2 \tag{7}$$

Let $\phi(x, y)$ and $\psi(x, y)$ be the solution of (7) given in the implicit form. Here we introduce two new coordinates α, β such that

$$\alpha = \phi(x, y) \quad \text{and} \quad \beta = \psi(x, y).$$

Therefore (1) reduces to the canonical form

$$u_{\alpha\beta} + \text{lower order terms} = 0. \tag{8}$$

2. **Parabolic Equation**: In this case we get only one root say ς, for the characteristic equation $q(\varsigma) = 0$.

Let $\phi(x, y)$ be the solution of $\dfrac{dy}{dx} = \varsigma$. Now choose $\alpha = \phi(x, y)$ and $\beta = \psi(x, y)$ can be choosen an independent function. Therefore (1) can be reduced to canonical form

$$u_{\beta\beta} + \text{lower order terms} = 0. \tag{9}$$

3. **Elliptic Equation**: Here we get complex roots where one of the roots is ς(say). Let the complex solution of $\dfrac{dy}{dx} = \varsigma$ be given by $\phi = \Phi + i\Psi$. Then $a = \Phi(x, y)$ and $\beta = \Psi(x, y)$ which is a real transformation. Therefore (1) is transformed into the canonical form

$$u_{\alpha\alpha} + u_{\beta\beta} + \text{lower order terms} = 0. \tag{10}$$

Example 1: The Tricomi equation $yu_{xx} + u_{yy} = 0$ is elliptic in the upper half plane, hyperbola in the lower half plane and parabola on the x-axis.

3 Second order linear PDE in n-variables

The general form of second order linear PDE in n variables is given by

$$\sum_{j,k=1}^{n} a_{jk}(x)u_{x_jx_k} + \sum_{j=1}^{n} u_{x_j} + c(x)u = f(x). \tag{11}$$

Let $x = (x_1, \dots, x_n)$. The principal part, characteristics and change of independent variable are given by

$$L = \sum_{j,k=1}^{n} a_{jk}(x)u_{x_jx_k}. \tag{12}$$

$$Q(\xi) = \sum_{j,k}^{n} a_{jk}\xi_j\xi_k, \quad \xi = (\xi_1, \dots, \xi_n). \tag{13}$$

$$y_j = \phi_j(x), \quad j = 1, \dots, n. \tag{14}$$

We obtain the transformed diagonal (canonical) form as

$$Lu = \sum_{j=1}^{n} \lambda_j u_{y_jy_j}, \quad \lambda_j = 0, \pm 1. \tag{15}$$

In order to achieve the characteristic equation, it is necessary to remove all the the coefficients $a_{jk},\ j \neq k$ from the principal part. Assuming the symmetry of coefficients, there are $\frac{1}{2}n(n-1)$ such coefficients to eliminate, but with only n functions as in (14).
If $n > 3$ then the canonical form cannot be achieved by means of change of independent variables.
If $n = 3$ then $\dfrac{1}{2}n(n-1)$, so that the diagonal form can be achieved, but there are no free variables left to satisfy the additional conditions that $\lambda = 0$ or ± 1.
Since the canonical form cannot be achieved, we fix an x. Hence the coefficients a_{jk} in Q are fixed.
From the linear algebra, $\exists$ an orthogonal linear transformation $\xi \to \eta$,

$$\eta_j = \sum_{k=1}^{n} \gamma_{jk}\xi_k. \tag{16}$$

So that $Q(\xi)$ transforms into the diagonal form,

$$\sum_{j=1}^{n} \lambda_j \eta_j^2. \tag{17}$$

Thus (11) is Elliptic if all the λ_j's in (17) are different from zero and of the same sign, Parabola if some of the λ_j's are zero and the remaining are of the same sign and Hyperbolic if none of the λ_j's are zero and all but one are of the same sign. If none of the λ_j's are zero but atleast two are positive and atleast two negative, the equation is called Ultra-Hyperbolic.

4 Linear second order PDE in two independent variables

Consider an equation of the form [1, 2, 3],

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u + g(x,y) = 0. \quad (18)$$

Let the transformations be

$$\xi = \xi(x,y) \quad \text{and} \quad \eta = \eta(x,y). \quad (19)$$

The Jacobian $J\begin{pmatrix} \xi & \eta \\ x & y \end{pmatrix}$ is given by $J = \dfrac{\partial(\xi,\eta)}{\partial(x,y)} = \xi_x\eta_y - \xi_y\eta_x \neq 0$.

The transformed equation is of the form

$$Au_{\xi\xi} + 2Bu_{\xi\eta} + Cu_{\eta\eta} + Du_\xi + Eu_\eta + Fu + G = 0, \quad (20)$$

where

$$\begin{aligned} A &= a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2, \\ B &= a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y, \\ C &= a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2, \\ D &= a\xi_{xx} + 2b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y, \\ E &= a\eta_{xx} + 2b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y, \\ F &= f, \quad G = g. \end{aligned} \quad (21)$$

Also A, B, C and a, b, c satisfy the relation $B^2 - 4AC = (b^2 - 4ac)(\xi_x\eta_y - \xi_y\eta_x)^2$. The sign of the expression $b^2 - 4ac$ remains invariant under the transformation. Based on the sign of $b^2 - 4ac$ we classify second order semilinear equations into Parabolic, Hyperbolic and Elliptic equations. A characteristic quadratic form is defined by

$$Q(l,m) = al^2 + 2blm + cm^2. \quad (22)$$

If

$$l = \lambda\xi_x + \mu\eta_x, \quad m = \lambda\xi_y + \mu\eta_y,$$

then (22) becomes

$$Q(l,m) = A\lambda^2 + 2B\lambda\mu + c\mu^2. \quad (23)$$

Depending on (23), there are three different cases for reduction to normal form.

Case 1: Let $Q(l,m) = A\lambda^2$, where $B = C = 0$. which arises when $B^2 - 4AC = 0$. In this case associated real symmetric matrix has one zero characteristic root $Q(l,m) = 1$ represents two parallel lines in l, m plane which will be a Parabola. The normal form is

$$Au_{\xi\xi} + Du_\xi + Eu_\eta + Fu + G = 0. \quad (24)$$

Case 2: Let $Q(l,m) = B\lambda\mu$, when $A = C = 0.$ and $B^2 - 4AC > 0$. In this case associated real symmetric matrix M has non zero characteristic roots of different signs. Here $Q(l,m) = 1$ represents a Hyperbola in l, m plane and the normal form is

$$2Bu_{\xi\eta} + Du_{\xi} + Eu_{\eta} + Fu + G = 0. \tag{25}$$

Case 3: Let $Q(l,m) = A(\lambda^2 + \mu^2)$ when $A = C, B = 0.$ and $B^2 - 4AC < 0$. In this case matrix M has non-zero characteristic of same sign. Here $Q(l,m) = 1$ represents an Ellipse in l,m plane. The normal form is

$$Au_{\xi\xi} + Cu_{\eta\eta} + Du_{\xi} + Eu_{\eta} + Fu + G = 0. \tag{26}$$

Also $\sigma = \xi + i\eta$ and $\tau = \xi - i\eta$ (or) $\xi = \dfrac{\sigma + \tau}{2}$ and $\eta = \dfrac{\sigma - \tau}{2i}$.

5 Method of reduction to normal form

From (25), the PDE of first order is given by

$$Q(\phi_x, \phi_y) = a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2 = 0. \tag{27}$$

In Case 3, solving for $\dfrac{-\phi_x}{\phi_y}$, we get two complex conjugate values

$$\frac{-\phi_x}{\phi_y} = \rho(x,y) \pm i\delta(x,y).$$

If we choose $\xi(x,y)$ and $\eta(x,y)$ such that

$$\frac{-(\xi \pm i\eta)x}{(\xi \pm i\eta)y} = \rho \pm i\delta.$$

Then

$$a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2$$

and

$$a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y.$$
$$A = C,\ B = 0.$$

To find the expression for ξ and η, let

$$\xi + i\eta = \sigma \quad \text{and} \quad \xi - i\eta = \tau.$$

Then σ, τ satisfy the equation

$$\frac{-\sigma_x}{\sigma_y} = \frac{dy}{dx} = \rho + i\delta, \quad \frac{-\tau_x}{\tau_y} = \frac{dy}{dx} = \rho - i\delta.$$

If we choose

$$\frac{-\xi_x}{\xi_y} = \frac{dy}{dx} = \rho_1, \quad \frac{-\eta_x}{\eta_y} = \frac{dy}{dx} = \rho_2,$$
$$A = C = 0.$$

In Case 2, (27) has distinct real roots ρ_1 and ρ_2 for $\dfrac{-\phi_x}{\phi_y}$.
In Case 1, (27) has equal real roots for $\dfrac{-\phi_x}{\phi_y}$ and $\dfrac{-\eta_x}{\eta_y} = \dfrac{dy}{dx} = \rho_1$ then $B = C = 0$,
Thus (18) reduces to (24) and ξ can be chosen arbitrarily such that ξ and η are functionally independent i.e, $\dfrac{\partial(\xi,\eta)}{\partial(x,y)} \neq 0$.

Example 2: Classify the given equation and transform it to the normal form.

$$u_{xx} + xu_{yy} = 0, \quad \forall x, y \tag{28}$$

$$b^2 - 4ac = -4x.$$

Case 1: If $x < 0$, (28) is hyperbolic equation.
We have the transformations

$$\xi(x,y) = \frac{3}{2}y + \sqrt[3]{-x} \quad \text{and} \quad \eta(x,y) = \frac{3}{2}y - \sqrt[3]{-x}. \tag{29}$$

Using (29) we can transform (28) to

$$u_{\xi\eta} + \frac{1}{6(\xi-\eta)}(u_\xi - u_\eta) = 0.$$

Case 2: If $x > 0$, (28) is Elliptic equation.
We have

$$\sigma(x,y) = \frac{3}{2}y - i\sqrt[3]{x} \quad \text{and} \quad \tau(x,y) = \frac{3}{2}y + i\sqrt[3]{x}.$$

By transforming σ, τ into $\xi - \eta$ plane, we get

$$\xi(x,y) = \frac{3}{2}y \quad \text{and} \quad \eta(x,y) = -\sqrt[3]{x}.$$

Using this transformation we get,

$$u_{\xi\xi} - y^2 u_{\eta\eta} + \frac{1}{3\eta}u_\eta = 0.$$

6 Linear second order PDEs in more than two independent variables

Consider the equation [1, 8, 9],

$$a_{\alpha\beta}u_{x_\alpha x_\beta} + b_\alpha u_{x_\alpha} + cu = d; \quad \alpha, \beta = 1, 2, \ldots, m \tag{30}$$

where $a_{\alpha\beta}$, b_α, c and d are functions of the independent variables $x_1, x_2, \ldots, x_m$ and $a_{\alpha\beta} = a_{\beta\alpha}$.
The associated real symmetric matrix will be $M = (a_{\alpha\beta})$.
Let us consider for $m = 3$,

$$a_{\alpha\beta}u_{x_\alpha x_\beta} = a_{11}u_{x_1x_1} + 2a_{21}u_{x_2x_1} + 2a_{31}u_{x_3x_1} + a_{22}u_{x_2x_2} + 2a_{32}u_{x_3x_2} + a_{33}u_{x_3x_3},$$

$$M = (a_{\alpha\beta}) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Consider the one-to-one transformation

$$\xi_\alpha = \xi_\alpha(x_1, x_2, \dots, x_m), \quad \alpha = 1, 2, \dots, m. \tag{31}$$

For $m = 3$,

$$\xi_\alpha = \xi_\alpha(x_1, x_2, x_3),$$

That is, $\xi_1 = \xi_1(x_1, x_2, x_3), \quad \xi_2 = \xi_2(x_1, x_2, x_3), \quad \xi_3 = \xi_3(x_1, x_2, x_3)$.
Using (31), (30) is transformed to

$$A_{\gamma\delta} u_{\xi_\gamma \xi_\delta} + D(\xi_1, \xi_2, \dots, \xi_m; u; u_{\xi_1}, \dots, u_{\xi_m}) = 0; \quad \gamma, \delta = 1, 2, \dots, m \tag{32}$$

where

$$A_{\gamma\delta} = a_{\alpha\beta}(\xi_\gamma)_{x_\alpha}(\xi_\delta)_{x_\beta}. \tag{33}$$

for $m = 3, \quad \gamma = \delta = 1$,

$$A_{11} = a_{11}\xi_{1_{x_1}}\xi_{1_{x_1}} + 2a_{12}\xi_{1_{x_1}}\xi_{1_{x_2}} + 2a_{13}\xi_{1_{x_1}}\xi_{1_{x_3}} + a_{22}\xi_{1_{x_2}}\xi_{1_{x_2}} + 2a_{23}\xi_{1_{x_2}}\xi_{1_{x_3}} + a_{33}\xi_{1_{x_3}}\xi_{1_{x_3}}.$$

At a given point, (30) can be reduced to canonical form by taking $a_{\alpha\beta}$ as constant. Let the point be chosen as the origin, then $\xi_\alpha = f_{\alpha\beta}x_\beta$ where $f_{\alpha\beta}$'s are constant. Thus (33) becomes $A_{\gamma\delta} = a_{\alpha\beta} f_{\gamma_\alpha} f_{\delta_\beta}$ and (32) has coefficients as constants. The same holds if (30) had constant coefficients.
The characteristic quadratic form $Q(\lambda)$ associated with (30) is

$$Q(\lambda) = a_{\alpha\beta}\lambda_\alpha\lambda_\beta = A_{\alpha\beta}\mu_\alpha\mu_\beta, \tag{34}$$

where λ's and μ's are related as follows
$\lambda_\alpha = \mu_\beta(\xi_\beta)_{x_\alpha} = \mu_\beta f_{\beta\alpha}; \quad \alpha, \beta = 1, 2, \dots, m$
For $m = 3$,

$$Q(\lambda) = a_{11}\lambda_1^2 + a_{22}\lambda_2^2 + a_{33}\lambda_3^2 + 2a_{12}\lambda_1\lambda_2 + 2a_{23}\lambda_3\lambda_2 + 2a_{13}\lambda_1\lambda_3.$$

Our aim is to transform $Q(\lambda)$ to principal axes so as to obtain the normal form of (30).
According to Principal Axes Theorem of Linear Algebra, any real symmetry matrix M is simultaneously similar to and congruent to a diagonal matrix D. Thus there exists an orthogonal matrix P, such that $D = PMP^{-1}$ is diagonal with special diagonal values. This means, by suitably choosing the constants $f_{\alpha\beta}$ and scaling ξ_α's, we have,
$A_{\alpha\beta} = 0 \quad \text{for} \quad \alpha \neq \beta$

$$A_{\alpha\alpha} = \begin{cases} +1 & \alpha = 1, 2, \dots, p \\ -1 & \alpha = p+1, \dots, r \\ 0 & \alpha = r+1, \dots, m. \end{cases}$$

depending on the fact that A has p positive eigen values, $(r - p)$ negative eigen values and $(m - r)$ zero eigen values.

Case 1: If all the eigen values of A are non-zero and of the same sign, the equation is of Elliptic type.

$$u_{\xi_1\xi_1} + u_{\xi_2\xi_2} + \dots + u_{\xi_m\xi_m} + D(\xi_1, \xi_2, \dots, \xi_m; u; u_{\xi_1}, \dots, u_{\xi_m}) = 0. \tag{35}$$

Case 2: If all the eigen values are non-zero and have the same sign, except precisely one of them, then the equation is of Normal Hyperbolic type.

$$u_{\xi_1\xi_1} - u_{\xi_2\xi_2} - \dots - u_{\xi_m\xi_m} + D(\xi_1, \xi_2, \dots, \xi_m; u; u_{\xi_1}, \dots, u_{\xi_m}) = 0. \tag{36}$$

Case 3: If all the eigen values are non-zero and atleast two eigenvalues are present with positive sign and atleast two with negative sign, then the equation is of Ultra-Hyperbolic type.
This situation can only occur when $n \geq 4$, the simplest case being the equation in four independent variables.

$$u_{\xi_1\xi_1} + u_{\xi_2\xi_2} = u_{\xi_3\xi_3} + u_{\xi_4\xi_4}. \tag{37}$$

Case 4: If any of the eigenvalues is zero, the equation is of Parabolic type.

$$u_t = u_{\xi_1\xi_1} + u_{\xi_2\xi_2} + \ldots + u_{\xi_{m-1}\xi_{m-1}}. \tag{38}$$

The Heat or Diffusion equation in $m-1$ space variables and one time variables is the best known Parabolic equation.

7 Method of reduction to normal form

We assign to the quadratic form $Q(\lambda)$, the first order Partial Differential Equation

$$Q_1(\phi) = a_{\alpha\beta}\phi_{x_\alpha}\phi_{x_\beta} = 0. \tag{39}$$

This is the characteristic equation of (30). In general, for equation (30) it is not necessary to find the orthogonal matrix P or the eigenvalues of M before classifying the equation. It is sufficient to express the quadratic form (34) as a sum of squares, which is merely the process of **'Completing the Squares'**. This is equivalent to a non-singular linear transformation of the variables.
By Sylvester's law of Inertia [4, 5], the number of positive and negative squares does not depend on the particular non-singular linear transformation used and is an invariant of the associated matrix. Thus the classification can be done depending on the number of positive and negative squares in the transformed quadratic form, as it will be the same as the number of positive and negative eigenvalues of the associated matrix.

Example 3: Classify the given equation and transform it to the normal form.

$$u_{xx} + 3u_{yy} + 84u_{zz} + 28u_{yz} + 16u_{zx} + 2u_{xy} = 0, \tag{40}$$

The associated matrix is $M = \begin{bmatrix} 1 & 1 & 8 \\ 1 & 3 & 14 \\ 8 & 14 & 84 \end{bmatrix}$.
Writing the Quadratic form $Q(\lambda)$ in λ_1, λ_2, λ_3, as

$$\begin{aligned} Q(\lambda) &= \lambda_1^2 + 3\lambda_2^2 + 84\lambda_3^2 + 2\lambda_1\lambda_2 + 16\lambda_1\lambda_3 + 28\lambda_2\lambda_3, \\ &= (\lambda_1 + \lambda_2 + 8\lambda_3)^2 + 2(\lambda_2 + 3\lambda_3)^2 + 2\lambda_3^2. \end{aligned}$$

Using the process of 'Completing the squares', we write $Q(\lambda)$ as the sum of squares. The equation is Elliptic as $Q(\lambda)$ can be expressed as the sum of three squares.

$$Q(\lambda) = (\lambda_1 + \lambda_2 + 8\lambda_3)^2 + [\sqrt{2}(\lambda_2 + 3\lambda_3)]^2 + (\sqrt{2}\lambda_3^2). \tag{41}$$

In matrix form,

$$Q(\lambda) = \begin{bmatrix} 1 & 1 & 8 \\ 0 & \sqrt{2} & 3\sqrt{2} \\ 0 & 0 & \sqrt{2} \end{bmatrix},$$

$$(Q^{-1})^T = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-5}{\sqrt{2}} & \frac{-3}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

We know that $D = PMP^{-1}$. Consider the transformation $Y = PX$ then equation (30) can be transformed to (32) with associated matrix N such that

$$N = PMP^T.$$

Let $N = D$ and $P = (Q^{-1})^T$.
Therefore the transformation which reduces (30) to the normal form is

$$Y = (Q^{-1})^T X.$$

The transformations can be written as

$$\xi_1 = x;\ \xi_2 = \frac{1}{\sqrt{2}}(y - x);\ \xi_3 = \frac{1}{\sqrt{2}}(z - 3y - 5x).$$

Therefore using these transformations (40) is transformed to

$$u_{\xi_1\xi_1} + u_{\xi_2\xi_2} + u_{\xi_3\xi_3}.$$

8 Conclusion

In this paper we have studied linear partial differential equations of two, three and more number of variables by reducing to canonical form. We have also studied the classification and types of PDEs by canonical transformation. We have demonstrated the method by using simple examples.

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An Analysis of Two-dimensional MHD Flow using DTM-Pade Approximation and Numerical Methods

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Abstract: This paper has focused on the investigation of flow of an electrically conducting fluid in presence of magnetic field for a viscous fluid on a stretching sheet embedded in a porous medium. The effects of magnetic field and permeability of the medium on the flow field are to be analyzed. The non-linear equation of the flow field has been solved by Differential transformation empowered by Pade approximates and Runge-Kutta method with shooting technique.The DTM-Pade proves to be an efficient method on comparison of the result with the numerical methods and we see the convergence for larger radius in case of DTM-Pade than Numerical method.

Keywords: MHD Flows, Porous medium, Boundary Layer, DTM-Pade approximation.

1 Intoduction

The study of the flow of electrically conducting fluid in presence of magnetic field is of importance in various of technology and engineering such as MHD power generation, MHD Flow meters and MHD pumps[2]. As an object moves through a fluid, or as a fluid moves past an object, the molecules of the fluid near the object are disturbed and move around the object. The magnitude of these forces depend on the shape of the object, the speed of the object, the mass of the fluid along the object and on two other important properties of the fluid; the viscosity, or stickiness, and the compressibility, or springiness of the fluid. Thus if a fluid flows in the presence of an obstacle, then the obstacle will experience two types of forces, drag force in the direction of motion of the fluid, lift force in a direction normal to the flow direction.

These two forces are produced by tangential and normal stresses. The shearing stress i.e., the drag due to tangential stress is called friction or skin friction or viscous drag. The drag due to normal stress is called pressure drag. Thus flows constrained by solid surfaces can typically be divided into two regions as below, Boundary Layer Region, Flow near a bounding surface with significant velocity with gradients normal to the solid body and shear stresses in this region are predominant. Potential Flow Region, Flows far from bounding surface with negligible velocity gradients, negligible shear stresses where inertia effects are important.

The boundary layer theory was first developed by Ludwig Prandtl in 1904. He gave a convincing explanation for motion of fluid around objects and this led to major advances in fluid dynamics. The detailed analysis of the flow within the boundary layer region is very important for many engineering problems and aerodynamics.

1.1 Prandtl Boundary Layer

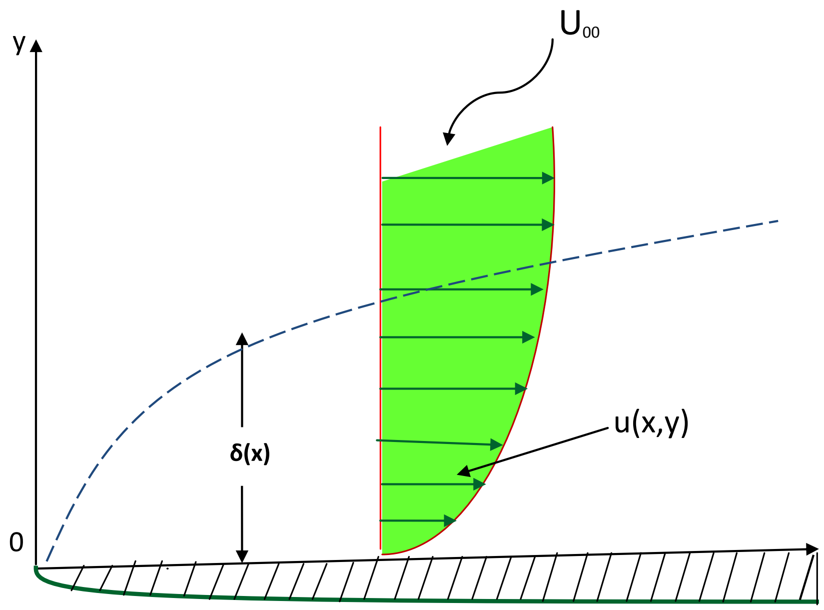

Figure 1: Schematic of the Prandtl Bounary layer of a viscous fluid.

Prandtl intoduced boundary layer theory in 1904 to understand the flow behaviour of a viscous fluid near a solid boundary. Prandtl gave the concept of a boundary layer in large Reynolds number flows and derived the boundary layer equations by simplifying the Navier-Stokes' equations to yield approximate solutions.

Consider the steady flow of a viscous incompressible fluid past a thin semi-infinite plate which is placed along the direction of a uniform stream of velocity U_∞ in Figure 1. Unlike an ideal (non-viscous) fluid flow, the fluid doesn't slide over the plate, but sticks to it. Since the plate is at rest, the fluid in contact with it will also be at rest.

As we move outward along the normal, the velocity of the fluid will gradually increase and at a distance far from the plate the full stream velocity U_∞ is attained.This is approached asymptotically. The transition from zero velocity at the plate to the full magnitude U_∞ takes place within a thin layer of fluid in contact with the plate. This is known as the Prandtl Boundary Layer.

2 Flow through a Porous Medium

Flow in a porous medium is a ordered flow in a disordered geometry. The transport process of fluid through a porous medium involves two substances: the fluid and the porous matrix and therefore, it will be characterized by specific properties of these substances.

A porous medium usually consists of a large number of interconnected pores each of which is saturated with the fluid. The exact form of the structure, however, is highly complicated and differs from medium to medium.

A porous medium may be either an aggregate of a larger number of particles such as sand or gravel or a solid containing many capillaries such as a porous rock, for example, limestone, or sponge. When the fluid percolates through a porous material, because of the complexity of microscopic flow in the pores, the actual path of an individual fluid particle cannot be followed analytically[3]. In that case, one has to consider the gross effect of the phenomena represented by a macroscopic view applied to the masses of fluid, large compared to the dimensions of the structure of the medium. The process can be described in terms of an equilibrium of forces. The driving force necessary to move a specific volume of fluid at a certain speed through a porous medium is in equilibrium with the resistance force generated by internal friction between the fluid and the pore structure. A more realistic approach to study dynamics of flow through porous media is under the assumption of continuum macroscopic phenomena. Usually the spatial averages are used to transfer properties of porous media from microscopic scale to macroscopic scale. Therefore, the definition of porosity and permeability is essential.

2.1 Porosity

Consider a point in a three dimensional flow region denoted by $x_i = (x_1, x_2, x_3)$ where x_i(i=1 to 3) are the co-ordinates of that point. Let V be a total volume containing both fluid and solid which may be either sphere or cube with center at x_i. Let V_v be the volume of voids. Then the porosity, ε, of such porous medium is defined as

$$\varepsilon = \frac{\text{void volume}}{\text{total volume}} = \frac{V_v}{V}(< 1)$$

If $V_v = V$ then it is the case for free fluid.

2.2 Permeability

Flow through a porous medium in the macroscopic continuum approach is described by the Darcy's law. For an an-isotropic porous medium Darcy's law can be expressed as

$$q_i = -k_{ij}\frac{\partial h}{\partial x_i} \tag{1}$$

where q_i(i=1,2,3) is the Darcy velocity, k_{ij}, a tensor, the hydraulic velocity of porous media and h is the water head at a point x_i which depends on the pressure P and density ρ and is a macroscopic quantity for an isotropic porous medium, kij reduces to a scalar k and then the Darcy law, given by (33) becomes

$$q_i = -k\frac{\partial h}{\partial x_i}$$

The hydraulic conductivity K of the porous medium depends on the properties of both solid and fluid aspect of porous media and is given by

$$K = \frac{k\rho g}{\mu}$$

where k is the permeability having dimension of $(\text{length})^2$, g is the gravity and ρ is the density and μ is the viscosity. Thus permeability measures quantitatively the ability of the porous medium to permit fluid flow [3].

3 Derivation of equation of continuity for an incompressible fluid

Let m be the mass of a fluid body and V be the volume at time t, then

$$m = \int_V \rho dV \tag{2}$$

where ρ is the density of fluid contained in an elementary volume dV.
The law of conservation of mass states that, "Mass remains unchanged during the motion of fluid", therefore

$$\frac{Dm}{Dt} = 0$$

Using the Reynolds transportation formula in above equation, we have

$$\int_V \left(\frac{D\rho}{Dt} + \rho\, div\, \vec{q} \right) dV = 0 \tag{3}$$

Since equation (3) is true for any arbitrary volume, we have

$$\frac{D\rho}{Dt} + \rho\, div\, \vec{q} = 0 \tag{4}$$

Equation (4) can be rewritten as

$$\frac{\partial \rho}{\partial t} + div(\rho \vec{q}) = 0 \tag{5}$$

Equation (5) gives equation of continuity for a compressible fluid [4].
For an incompressible fluid, we have

$$div(\vec{q}) = 0 \tag{6}$$

For a two-dimensional flow, $\vec{q} = u\hat{i} + v\hat{j}$, hence the equation (5) becomes,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{7}$$

Equation (7) is the requires continuity equation for an incompressible electrically conducting fluid.

4 Derivation of equation of momentum for an electrically conducting incompressible fluid

Consider a fluid element of an electrically conducting incompressible fluid enclosing a volume V bounded by a surface S.
The linear momentum $\vec{p}$ of the fluid is defined by

$$\vec{p} = \int_V \rho\, \vec{q} dV \tag{8}$$

where ρ is the density and $\vec{q}$ is the velocity of the fluid at that point.
The resultant force $f(\vec{r})$ acting on the fluid is given by

$$f(\vec{r}) = \int_V \rho\vec{b}\,dV + \int_V \vec{J}\times\vec{B}\,dV + \int_S \vec{s}dS \tag{9}$$

where $\vec{b}$ is the body force per unit mass, $\vec{s}$ is the surface force per unit area called stress vector, and $\vec{J}\times\vec{B}$ gives the Lorentz force.
The law of conservation of mass states that, "The material time rate of change of linear momentum is equal to the total force", therefore

$$\frac{D\vec{p}}{Dt} = f(\vec{r}) \tag{10}$$

From Cauchy law of Stress,

$$\vec{s} = \underset{\sim}{T}\cdot\hat{n} \tag{11}$$

where $\underset{\sim}{T}$ is the Stress tensor and $\hat{n}$ is the unit outward normal
Using equations (8), (9) and (11) in equation (10), we get

$$\int_V \left[\rho\frac{D\vec{q}}{Dt} - \rho\vec{b} - div\underset{\sim}{T}^T - \vec{J}\times\vec{B}\right] dV = 0 \tag{12}$$

Since equation (12) is true for any arbitrary volume, we have

$$\rho\frac{D\vec{q}}{Dt} - \rho\vec{b} - div\underset{\sim}{T} - \vec{J}\times\vec{B} \tag{13}$$

where $T = \underset{\sim}{T}^T$.
The stress tensor for viscous fluid is given by,

$$\underset{\sim}{T} = (-P + \lambda div\vec{q})\,\underset{\sim}{I} + \mu(\nabla\vec{q} + \nabla\vec{q}^T) \tag{14}$$

Taking divergence on both sides of equation , we get

$$div\,\underset{\sim}{T} = -\nabla P + \mu\nabla^2\vec{q} \tag{15}$$

Substituting equation (15) in (13), we get

$$\rho\left[\frac{\partial\vec{q}}{\partial t} + (\vec{q}\cdot\nabla)\vec{q}\right] = -\nabla P + \mu\nabla^2\vec{q} + \vec{J}\times\vec{B} \tag{16}$$

Equation (16) gives the required momentum equation for an electrically conducting incompressible fluid.
Further for steady flow $\dfrac{\partial\vec{q}}{\partial t} = 0$, therefore, equation (16) becomes,

$$\rho\left[(\vec{q}\cdot\nabla)\vec{q}\right] = -\nabla P + \mu\nabla^2\vec{q} + \vec{J}\times\vec{B} \tag{17}$$

Using Darcy's law, we have

$$\nabla P = \frac{\mu\vec{q}}{k_p(x)} \tag{18}$$

where $\vec{q}$ is fluid velocity, $k_p(x)$ is the variable velocity and μ is the fluid viscosity.
Substituting equation (18) in (17), we get

$$(\vec{q}\cdot\nabla)\vec{q} = -\frac{\nu\vec{q}}{k_p(x)} + \nu\nabla^2\vec{q} + \frac{1}{\rho}\vec{J}\times\vec{B} \tag{19}$$

where $\nu = \dfrac{\mu}{\rho}$ is kinematic viscosity.
The equation along flow direction is given as,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B^2(x)u}{\rho} - \frac{\nu u}{K_p(x)} \tag{20}$$

5 An analysis using DTM-Pade method for a flow problem with applied magnetic field

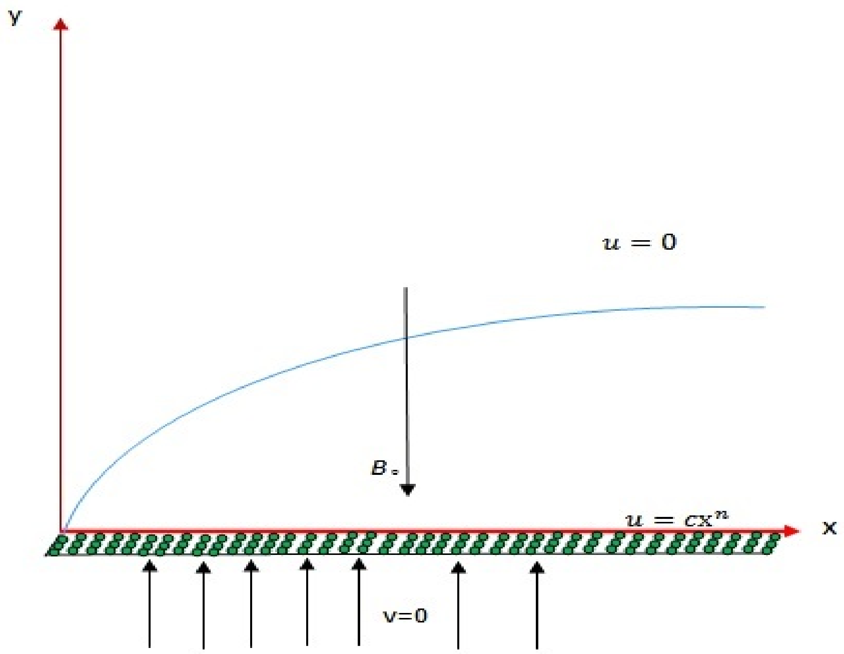

Figure 2: Schematic of the geometry of Two-Dimensional viscous incompressible electrically conducting fluid flow with applied magnetic field

Consider a steady two dimensional MHD boundary layer flow of a viscous incompressible elecrically conducting fluid over a thin flat stretching plate embedded in a porous medium which is placed in the direction of flow in Figure 2. Let the origin of the co-ordinate be at leading edge of the plate, the x-axis be the direction of the uniform stream and the y-axis normal to the plate. A transverse magnetic field of strength B_0 has been applied perpendicular to the plate.

The Prandtl boundary layer-Darcian flow equations subject to above consideration are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{21}$$

$$u\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \sigma\frac{B^2(x)}{\rho}u - \frac{\nu u}{k_p(x)} \tag{22}$$

and the corresponding boundary conditions are

$$\begin{aligned} u(x,0) = cx^n,\ v(x,0) \\ u(x,y) \to 0,\ y \to \infty \end{aligned} \tag{23}$$

where u and v are the components of the velocity in the x and y directions, and ν is the kinematic viscosity.
The stream function $\psi(x,y)$ is introduced such that

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = \frac{-\partial \psi}{\partial x} \tag{24}$$

Now, we will convert the partial differential equation (22) into an ordinary differential equation. In accordance with the procedure of the law of similarity, let the velocity profile be

$$u = U_\infty F(\eta) \tag{25}$$

where

$$\eta(x,y) = \sqrt{\frac{c(n+1)}{2\nu}}\, x^{\frac{n-1}{2}}\, y \tag{26}$$

is the similarity variable.
Using equation (24),(25) and (26), the stream function $\psi(x,y)$ is given by

$$\psi = \int u dy = \sqrt{\frac{2\nu c}{n+1}}\, x^{\frac{n+1}{2}} \int F(\eta)d\eta = \sqrt{\frac{2\nu c}{n+1}}\, x^{\frac{n+1}{2}}\, f(\eta) \tag{27}$$

where $f(\eta) = \int F(\eta)\, d\eta$.
The velocity components and their derivatives are given by

$$u = \frac{\partial \psi}{\partial y} = cx^n f'(\eta), \tag{28}$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{c(n-1)}{2}x^{n-1}\, y\, f'(\eta) - \sqrt{\frac{2\nu c}{n+1}}\left(\frac{n+1}{2}\right) x^{\frac{n-1}{2}}\, f(\eta) \tag{29}$$

$$\frac{\partial u}{\partial x} = \frac{\partial^2 \psi}{\partial x \partial y} = c\left\{ nx^{n-1}f'(\eta) + \sqrt{\frac{c(n+1)}{2\nu}}\left(\frac{n-1}{2}\right) x^{\frac{3n-3}{2}}\, yf''(\eta)\right\} \tag{30}$$

$$\frac{\partial u}{\partial y} = \frac{\partial^2 \psi}{\partial y^2} = c\sqrt{\frac{c(n+1)}{2\nu}}x^n x^{\frac{n-1}{2}}\, f''(\eta) \tag{31}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^3 \psi}{\partial y^3} = \frac{c^2(n+1)}{2\nu}x^{2n-1}f'''(\eta) \tag{32}$$

Substituting equations (28)-(32) in equation (22), we get

$$f''' + f\, f'' - \beta f'^2 - \left(M + \frac{1}{K_p}\right) f' = 0 \tag{33}$$

and the corresponding boundary conditions are

$$f(0) = 0,\ f'(0) = 1,\ f'(\infty) = 0$$

$$\beta = \frac{2n}{n+1}, M = \frac{2\sigma B_0^2}{\rho c(1+n)}, \frac{1}{K_p} = \frac{2\nu}{c(1+n)k_p'} \tag{34}$$

M is the magnetic parameter, K_p is the permeability parameter and β is the power index[2]. Now equation (33) along with the boundary conditions $f(0) = 0, f'(0) = 1, f'(\infty) = 0$ are converted into IVP using Shooting Method [1] and value of $f''(0)$ is evaluated as shown in table below:

Table 1

β	M	k_p	$f''(0)$
0.5	3	100	-1.918464
0.5	4	100	-2.163251
1	3	100	-2.002498
2	3	50	-2.163387
3	1	100	-1.822783

We plot graphs of $f'(\eta)$ verses t using 5^{th} order Runge-Kutta-Felhberg method[1]. This method gives the required convergence of the variable which is appreciable.

5.1 Analytical approximation by means of the DTM-Pade

Consider a function $w(x)$ which is analytic in a domain T and let $x = x_0$ represent any point in the domain T. The function $w(x)$ is then represented by a power series whose center is located at x_0. The differential transform of the k^{th} derivative of a function $w(x)$ is given by

$$W(k) = \frac{1}{k!}\left[\frac{d^k w(x)}{dx^k}\right]_{x=x_0} \tag{35}$$

where $w(x)$ is the original function and $W(k)$ the transformed function. The inverse transformation is defined as follows

$$w(x) = \sum_{k=0}^{\infty}(x - x_0)^k W(k) \tag{36}$$

Combining equations (35) and (36), we obtain:

$$w(x) = \sum_{k=0}^{\infty} \frac{(x-x_0)^k}{k!}\left[\frac{d^k w(x)}{dx^k}\right] \tag{37}$$

From equation (37), one can analyze that the concept of DTM is derived from Taylor series expansion. Assuming $x_0 = 0$, we have documented operations for differential transformed functions about the point $x = 0$ as shown in Table 2 given in Appendix-A [2].

Taking differential transform of equation (34) by using the related definitions given in Table 2 in Appendix-A, we obtain

$$\begin{aligned}(k+1)(k+2)(k+3)F(k+3) + \sum_{m=0}^{k} F(m)(k-m+1)(k-m+2)F(k-m+2)\\ -\beta\sum_{m=0}^{k}(m+1)F(m+1)(k-m+1)F(k-m+1) - \left(M + \frac{1}{K_p}\right)(k+1)F(k+1) = 0\end{aligned} \tag{38}$$

and the corresponding transformed boundary conditions are

$$F(0) = 0, F(1) = 1, F(2) = \alpha \tag{39}$$

$$\begin{aligned} f(\eta) &= \sum_{k=0}^{\infty} F(k)\eta^k \\ &= F(0) + F(1)\eta + F(2)\eta^2 + F(3)\eta^3 + F(4)\eta^4 + F(5)\eta^5 + F(6)\eta^6 \\ &+ F(7)\eta^7 + F(8)\eta^8 + F(9)\eta^9 + \dots \end{aligned} \tag{40}$$

where

$$F(3) = \frac{1}{6}\beta + \frac{1}{6}(M + 1/K_p)$$
$$F(4) = -\frac{\alpha}{12} + \frac{\alpha\beta}{6} + \frac{\alpha(M + 1/K_p)}{12}$$
$$F(5) = \frac{-\beta}{60} - \frac{M + 1/K_p}{60} - \frac{\alpha^2}{30} + \frac{\beta^2}{60} - \frac{(M + 1/K_p)\beta}{40} + \frac{\alpha^2\beta}{15} + \frac{(M + 1/K_p)^2}{120}$$

$$F(6) = \frac{\alpha\beta}{36K_p} + \frac{M\alpha}{180K_p} - \frac{\alpha}{45K_p} + \frac{\alpha}{360{K_p}^2} + \frac{\alpha\beta^2}{36} + \frac{M\alpha\beta}{36} - \frac{\alpha\beta}{30} + \frac{M^2\alpha}{360} - \frac{M\alpha}{45} + \frac{\alpha}{120}$$

$$F(7) = \frac{\beta^2}{252K_p} + \frac{\alpha^2\,\beta}{126K_p} + \frac{11M\,\beta}{2520K_p} - \frac{13\beta}{2520K_p} - \frac{2\alpha^2}{315K_p} + \frac{M^2}{1680K_p} - \frac{M}{252K_p} + \frac{1}{630K_p}$$

$$+\frac{11\beta}{5040{K_p}^2} + \frac{M}{1680{K_p}^2} - \frac{1}{504{K_p}^2} + \frac{1}{5040{K_p}^3} + \frac{\beta^3}{504} + \frac{\alpha^2\,\beta^2}{63} + \frac{M\,\beta^2}{252}$$

$$-\frac{\beta^2}{315} + \frac{M\,\alpha^2\,\beta}{126} - \frac{8\alpha^2\,\beta}{315} + \frac{11M^2\,\beta}{5040} - \frac{13M\,\beta}{2520} + \frac{\beta}{630}$$

$$+\frac{11\alpha^2}{1260} + \frac{M^3}{5040} - \frac{M^2}{504} + \frac{M}{630}$$

$$F(8) = \frac{\alpha\beta^2}{168K_p} + \frac{M\alpha\beta}{240K_p} - \frac{7\alpha\beta}{720K_p} + \frac{M^2\alpha}{6720K_p} - \frac{13M\alpha}{3360K_p} + \frac{\alpha}{252K_p} + \frac{\alpha\beta}{480{K_p}^2}$$

$$+\frac{M\alpha}{6720{K_p}^2} - \frac{13\alpha}{6720{K_p}^2} + \frac{\alpha}{20160{K_p}^3} + \frac{\alpha\beta^3}{252} + \frac{\alpha^3\,\beta^2}{252} + \frac{M\alpha\beta^2}{168} -$$

$$\frac{83\alpha\beta^2}{10080} - \frac{2\alpha^3\,\beta}{315} + \frac{M^2\alpha\beta}{480} - \frac{7M\alpha\beta}{720} + \frac{103\alpha\beta}{20160} + \frac{11\alpha^3}{5040}$$

$$+\frac{M^3\alpha}{20160} - \frac{13M^2\alpha}{6720} + \frac{M\alpha}{252} - \frac{\alpha}{1344}$$

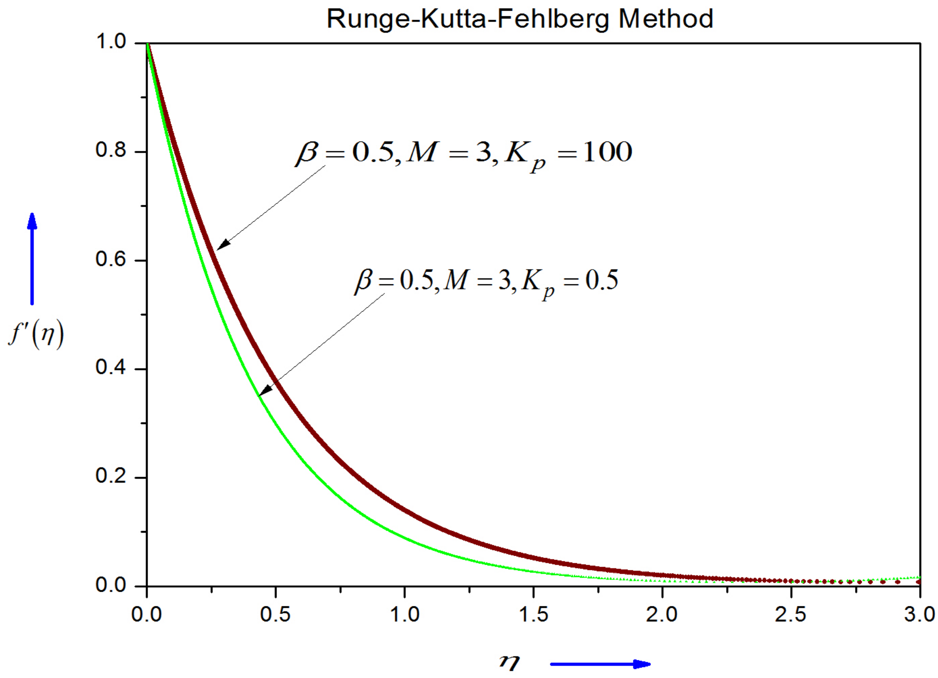

Figure 3: The Velocity Profile of $f(\eta)'$ using Runge-kutta-Felhberg Method

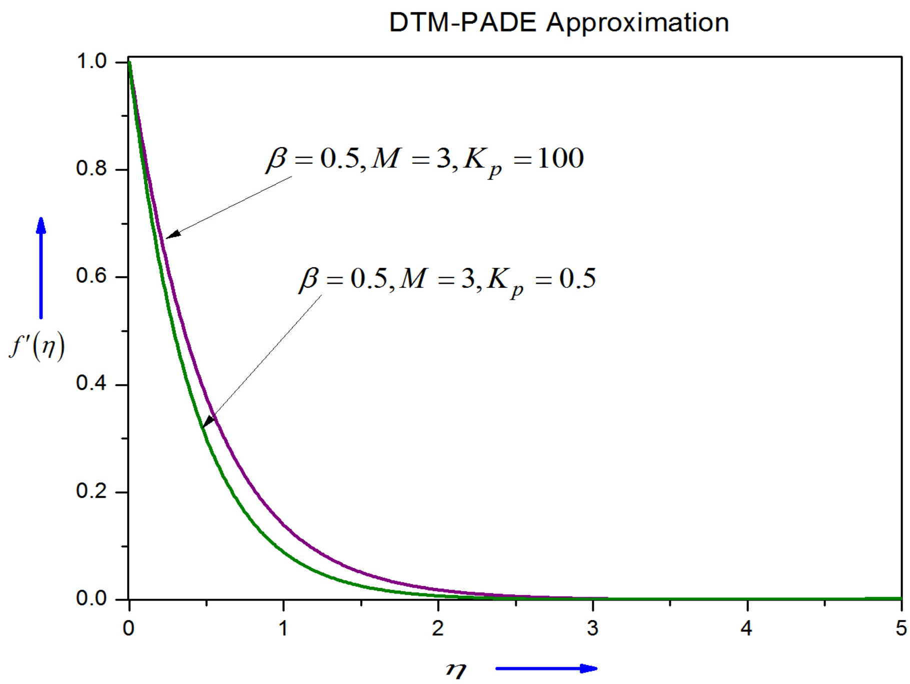

Figure 4: The Velocity Profile of $f(\eta)'$ using DTM-Pade Approximation for Prandtl boundary layer

6 Results and Discussions

We have plotted the velocity for the different power index (β) and the graphs in Figures 3 and 4 shows that the velocity decreases asymptotically with the progress of the fluid flow. The skin friction obtained by Runge-kutta method using shooting technique. The $f'(\eta)$ has been approximated using $[7/7]$ Pade rational function approximation. The shear stress is increased due to permeability.The numerical solution obtained has a faster convergence than DTM which is observed in graph in Figure 3 but DTM has the longer convergence.

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Applications of Method of Characteristics for System of Equations

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Abstract: In this paper we outline Jacobi method to solve nonlinear partial differential equation involving more than two independent variables. This method is similar to Charpit's method for two variables. We have also extended to Hamilton Jacobi equations which are for $(n+1)$ independent variables. Methods are outlined with examples. Jacobi theory is also outlined with example which is useful for solving ODE by Converting into PDE.

Keywords: Nonlinear partial differential equations, Jacobi method, Poisson bracket, Lagrange bracket

1 Introduction

Consider a partial differential equation [4, 1]

$$F(x, y, \ldots, u, u_x, u_y, \ldots, u_{xx}, u_{xy}, \ldots) = 0, \tag{1}$$

where F is a function of the variables $x, y, \ldots u, u_x, u_y, \ldots, u_{xx}, u_{xy}, \ldots$ A function $u(x, y, \ldots)$ of the independent variables $x, y, \ldots$ is sought such that (1) is identically satisfied in these independent variables, if $u(x, y, \ldots)$ and its partial derivatives $u_x, u_y, \ldots, u_{xx}, u_{xy}, \ldots$ are substituted in F. Such a function $u(x, y, \ldots)$ is called a solution of the partial differential equation (1).

1.1 The Complete Integral

Consider the differential equation

$$F(x, y, u, p, q) = 0. \tag{2}$$

For a function $u(x, y)$ of two independent variables have the solution $u = \phi(x, y, a, b)$ which depends on two parameters a, b. If u does not appear explicitly in F then the one parameter family of solution $u = \phi(x, y, a)$ leads immediately to a family $u = \phi(x, y, a) + b$ which depends on two parameters. A two parameter family of solutions is called complete integral of differential equation.
The rank of the matrix,

$$M = \begin{pmatrix} \phi_\alpha & \phi_{xa} & \phi_{ya} \\ \phi_\beta & \phi_{xa} & \phi_{ya} \end{pmatrix}$$

will be 2^1 if $D = \phi_{x\alpha}\phi_{yb} - \phi_{xb}\phi_{ya} \neq 0$.
The basic idea behind Complete integral is the formation of envelopes or by differentiation and elimination processes, one may obtain from a complete integral and arbitrary function.

1.2 Simultaneous Differential Equations of the First Order and First Degree in Three Variables

Consider the system of simultaneous differential equations of the first order and first degree of the type [3],

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \ldots, t), \quad i = 1, 2, \ldots, n. \tag{3}$$

Here we find the n functions x_i, which depends on t and initial conditions and which satisfy the set of (3) identically in t. For example, a differential equation of the n^{th} order

$$\frac{d^n x}{dt^n} = f\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \ldots, \frac{d^{n-1}x}{dt^{n-1}}\right), \tag{4}$$

can be written as

$$\frac{dx}{dt} = y_1, \quad \frac{dy_1}{dt} = y_2, \quad \frac{dy_2}{dt} = y_3, \ \ldots, \frac{dy_{n-1}}{dt} = f(t, x, y_1, y_2, \ldots, y_{n-1}). \tag{5}$$

System of differential equations of kind (3) arises in analytical mechanics. In Hamiltonian form the equations of motion of a dynamical system of n degrees of freedom are.

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \ i = 1, 2, \ldots, n \tag{6}$$

where $H(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n, t)$ is the Hamiltonian function of the system. These Hamiltonian equations of motion form a set of (3) for the $2n$ unknown functions $q_1, q_2, \ldots, q_n$, $p_1, p_2, \ldots, p_n$ the solution of which provides a description of the properties of the dynamical system at any time t. If the dynamical system possesses only one degree of freedom i.e., if its configuration at ant time is uniquely specified by a single coordinate q then the equation of motion reduce to the simple form

$$\frac{dp}{dt} = \frac{\partial H}{\partial q}, \quad \frac{dq}{dt} = \frac{\partial H}{\partial p}, \tag{7}$$

where $H(p, q, t)$ is the Hamiltonian of the system.

1.3 Poisson Bracket

Poisson bracket is an operator which takes two functions of phase space and time and produces a new function [9, 7].

Let us consider two functions $F_1(q_i, p_i, t)$ and $F_2(q_i, p_i, t)$, then the Poisson bracket is defined as

$$\{F_1, F_2\} = \sum_{i=1}^{n} \left(\frac{\partial F_1}{\partial q_i}\frac{\partial F_2}{\partial p_i} - \frac{\partial F_1}{\partial p_i}\frac{\partial F_2}{\partial q_i}\right). \tag{8}$$

Example 1: Let $F_1(q, p) = p - q, \quad F_2(q, p) = \sin q$. Then the Poisson bracket is given by

$$\{F_1, F_2\} = \left(\frac{\partial F_1}{\partial q}\frac{\partial F_2}{\partial p} - \frac{\partial F_1}{\partial p}\frac{\partial F_2}{\partial q}\right),$$

$$\{F_1, F_2\} = -\cos q.$$

1.4 Lagrange Bracket

If $(q_1, q_2, \ldots, q_n; p_1, p_2, \ldots, p_n)$ is a system of canonical coordinates in a phase space and if F_1, F_2 be the functions of those variables then Lagrange bracket is defined as

$$\{F, G\} = \sum_i \left(\frac{\partial q_i}{\partial F_1} \frac{\partial p_i}{\partial F_2} - \frac{\partial p_i}{\partial F_2} \frac{\partial q_i}{\partial F_2} \right). \tag{9}$$

2 Mathematical Techniques: Methodology

2.1 Jacobi's Method

Jacobi method is used for solving partial differential equations involving three or more independent variables [5, 8]. This method is almost similar to Charpit's method for two independent variables.
Consider a partial differential equation

$$f(x_1, x_2, x_3, p_1, p_2, p_3) = 0, \tag{10}$$

where x_1, x_2, x_3 are independent variables, z is the dependent variable and $p_1 = \dfrac{\partial z}{\partial x_1},\ p_2 = \dfrac{\partial z}{\partial x_2},\ p_3 = \dfrac{\partial z}{\partial x_3}$. Note that the dependent variable z does not appear in the partial differential equation (10). In Jacobi method the main idea is to get two additional partial differential equations of the first order.

$$F_1(x_1, x_2, x_3, p_1, p_2, p_3) = a_1, \tag{11}$$

$$F_2(x_1, x_2, x_3, p_1, p_2, p_3) = a_2, \tag{12}$$

where a_1 and a_2 are two arbitrary constants. To find p_1, p_2, p_3 solve the equations (10), (11), and (12) in terms of x_1, x_2, x_3 and substituted in,

$$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3. \tag{13}$$

To make equation (13) as integrable, we consider the conditions that are,

$$\frac{\partial p_2}{\partial x_1} = \frac{\partial p_1}{\partial x_2}, \quad \frac{\partial p_3}{\partial x_2} = \frac{\partial p_2}{\partial x_3}, \quad \frac{\partial p_1}{\partial x_3} = \frac{\partial p_3}{\partial x_1}. \tag{14}$$

Then differentiating equations (10) and (11) partially with respect to x_1, we get

$$\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial p_1}\frac{\partial p_1}{\partial x_1} + \frac{\partial f}{\partial p_2}\frac{\partial p_2}{\partial x_1} + \frac{\partial f}{\partial p_3}\frac{\partial p_3}{\partial x_1} = 0, \tag{15}$$

$$\frac{\partial F_1}{\partial x_1} + \frac{\partial F_1}{\partial p_1}\frac{\partial p_1}{\partial x_1} + \frac{\partial F_1}{\partial p_2}\frac{\partial p_2}{\partial x_1} + \frac{\partial F_1}{\partial p_3}\frac{\partial p_3}{\partial x_1} = 0. \tag{16}$$

Eliminating $\dfrac{\partial p_1}{\partial x_1}$ from (15) and (16)

$$\left(\frac{\partial f}{\partial x_1}\frac{\partial F_1}{\partial p_1} - \frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial x_1} \right) + \left(\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_1} - \frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial p_2} \right)\frac{\partial p_2}{\partial x_1} + \left(\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial p_1} - \frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial p_3} \right)\frac{\partial p_3}{\partial x_1} = 0. \tag{17}$$

Similarly differentiating equations (10) and (11) partially with respect to x_2, we get

$$\frac{\partial f}{\partial x_2}+\frac{\partial f}{\partial p_1}\frac{\partial p_1}{\partial x_2}+\frac{\partial f}{\partial p_2}\frac{\partial p_2}{\partial x_2}+\frac{\partial f}{\partial p_3}\frac{\partial p_3}{\partial x_2}=0, \tag{18}$$

$$\frac{\partial F_1}{\partial x_2}+\frac{\partial F_1}{\partial p_1}\frac{\partial p_1}{\partial x_2}+\frac{\partial F_1}{\partial p_2}\frac{\partial p_2}{\partial x_2}+\frac{\partial F_1}{\partial p_3}\frac{\partial p_3}{\partial x_2}=0. \tag{19}$$

Eliminating $\frac{\partial p_2}{\partial x_2}$ from (18) and (19), we have

$$\left(\frac{\partial f}{\partial x_2}\frac{\partial F_1}{\partial p_2}-\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial x_2}\right)+\left(\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial p_2}-\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_1}\right)\frac{\partial p_1}{\partial x_2}+\left(\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial p_2}-\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_3}\right)\frac{\partial p_3}{\partial x_2}=0. \tag{20}$$

Again differentiating equations (10) and (11) partially with respect to x_3, we get

$$\frac{\partial f}{\partial x_3}+\frac{\partial f}{\partial p_1}\frac{\partial p_1}{\partial x_3}+\frac{\partial f}{\partial p_2}\frac{\partial p_2}{\partial x_3}+\frac{\partial f}{\partial p_3}\frac{\partial p_3}{\partial x_3}=0, \tag{21}$$

$$\frac{\partial F_1}{\partial x_3}+\frac{\partial F_1}{\partial p_1}\frac{\partial p_1}{\partial x_3}+\frac{\partial F_1}{\partial p_2}\frac{\partial p_2}{\partial x_3}+\frac{\partial F_1}{\partial p_3}\frac{\partial p_3}{\partial x_3}=0. \tag{22}$$

Eliminating $\frac{\partial p_3}{\partial x_3}$ from (21) and (22), we have

$$\left(\frac{\partial f}{\partial x_3}\frac{\partial F_1}{\partial p_3}-\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial x_3}\right)+\left(\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial p_3}-\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial p_1}\right)\frac{\partial p_1}{\partial x_3}+\left(\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_3}-\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial p_2}\right)\frac{\partial p_2}{\partial x_3}=0. \tag{23}$$

Adding the (17), (20) and (23) and also use the relation (14), we get

$$\left(\frac{\partial f}{\partial x_1}\frac{\partial F_1}{\partial p_1}-\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial x_1}\right)+\left(\frac{\partial f}{\partial x_2}\frac{\partial F_1}{\partial p_2}-\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial x_2}\right)+\left(\frac{\partial f}{\partial x_3}\frac{\partial F_1}{\partial p_3}-\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial x_3}\right)=0. \tag{24}$$

The L. H. S of equation (24) is denoted by (f, F_1)

$$(f,F_1)=\sum_{r=1}^{3}\left(\frac{\partial f}{\partial x_r}\frac{\partial F_1}{\partial p_r}-\frac{\partial f}{\partial p_r}\frac{\partial F_1}{\partial x_r}\right)=0. \tag{25}$$

Similarly consider the equations (10) and (12) and differentiate partially with respect to x_1, we get

$$\left.\begin{aligned}\frac{\partial f}{\partial x_1}+\frac{\partial f}{\partial p_1}\frac{\partial p_1}{\partial x_1}+\frac{\partial f}{\partial p_2}\frac{\partial p_2}{\partial x_1}+\frac{\partial f}{\partial p_3}\frac{\partial p_3}{\partial x_1}=0,\\ \frac{\partial F_2}{\partial x_1}+\frac{\partial F_2}{\partial p_1}\frac{\partial p_1}{\partial x_1}+\frac{\partial F_2}{\partial p_2}\frac{\partial p_2}{\partial x_1}+\frac{\partial F_2}{\partial p_3}\frac{\partial p_3}{\partial x_1}=0.\end{aligned}\right\} \tag{26}$$

Eliminating $\frac{\partial p_1}{\partial x_1}$ from (26), we have

$$\left(\frac{\partial f}{\partial x_1}\frac{\partial F_2}{\partial p_1}-\frac{\partial f}{\partial p_1}\frac{\partial F_2}{\partial x_1}\right)+\left(\frac{\partial f}{\partial p_2}\frac{\partial F_2}{\partial p_1}-\frac{\partial f}{\partial p_1}\frac{\partial F_2}{\partial p_2}\right)\frac{\partial p_2}{\partial x_1}+\left(\frac{\partial f}{\partial p_3}\frac{\partial F_2}{\partial p_1}-\frac{\partial f}{\partial p_1}\frac{\partial F_2}{\partial p_3}\right)\frac{\partial p_3}{\partial x_1}=0. \tag{27}$$

Differentiating equations (10) and (12) partially with respect to x_2, we get

$$\left.\begin{aligned}\frac{\partial f}{\partial x_2}+\frac{\partial f}{\partial p_1}\frac{\partial p_1}{\partial x_2}+\frac{\partial f}{\partial p_2}\frac{\partial p_2}{\partial x_2}+\frac{\partial f}{\partial p_3}\frac{\partial p_3}{\partial x_2}=0,\\ \frac{\partial F_2}{\partial x_2}+\frac{\partial F_2}{\partial p_1}\frac{\partial p_1}{\partial x_2}+\frac{\partial F_2}{\partial p_2}\frac{\partial p_2}{\partial x_2}+\frac{\partial F_2}{\partial p_3}\frac{\partial p_3}{\partial x_2}=0.\end{aligned}\right\} \tag{28}$$

Eliminating $\frac{\partial p_2}{\partial x_2}$ from (28), we have

$$\left(\frac{\partial f}{\partial x_2}\frac{\partial F_2}{\partial p_2}-\frac{\partial f}{\partial p_2}\frac{\partial F_2}{\partial x_2}\right)+\left(\frac{\partial f}{\partial p_1}\frac{\partial F_2}{\partial p_2}-\frac{\partial f}{\partial p_2}\frac{\partial F_2}{\partial p_1}\right)\frac{\partial p_1}{\partial x_2}+\left(\frac{\partial f}{\partial p_3}\frac{\partial F_2}{\partial p_2}-\frac{\partial f}{\partial p_2}\frac{\partial F_2}{\partial p_3}\right)\frac{\partial p_3}{\partial x_2}=0. \tag{29}$$

Differentiating equations (10) and (12) partially with respect to x_3, we get

$$\left.\begin{aligned}\frac{\partial f}{\partial x_3}+\frac{\partial f}{\partial p_1}\frac{\partial p_1}{\partial x_3}+\frac{\partial f}{\partial p_2}\frac{\partial p_2}{\partial x_3}+\frac{\partial f}{\partial p_3}\frac{\partial p_3}{\partial x_3}=0,\\ \frac{\partial F_2}{\partial x_3}+\frac{\partial F_2}{\partial p_1}\frac{\partial p_1}{\partial x_3}+\frac{\partial F_2}{\partial p_2}\frac{\partial p_2}{\partial x_3}+\frac{\partial F_2}{\partial p_3}\frac{\partial p_3}{\partial x_3}=0.\end{aligned}\right\} \tag{30}$$

Eliminating $\frac{\partial p_3}{\partial x_3}$ from (30), we have

$$\left(\frac{\partial f}{\partial x_3}\frac{\partial F_2}{\partial p_3}-\frac{\partial f}{\partial p_3}\frac{\partial F_2}{\partial x_3}\right)+\left(\frac{\partial f}{\partial p_1}\frac{\partial F_2}{\partial p_3}-\frac{\partial f}{\partial p_3}\frac{\partial F_2}{\partial p_1}\right)\frac{\partial p_1}{\partial x_3}+\left(\frac{\partial f}{\partial p_2}\frac{\partial F_2}{\partial p_3}-\frac{\partial f}{\partial p_3}\frac{\partial F_2}{\partial p_2}\right)\frac{\partial p_2}{\partial x_3}=0. \tag{31}$$

Adding the equations (27), (29) and (31) and also use the relation (14), we get

$$\left(\frac{\partial f}{\partial x_1}\frac{\partial F_2}{\partial p_1}-\frac{\partial f}{\partial p_1}\frac{\partial F_2}{\partial x_1}\right)+\left(\frac{\partial f}{\partial x_2}\frac{\partial F_2}{\partial p_2}-\frac{\partial f}{\partial p_2}\frac{\partial F_2}{\partial x_2}\right)+\left(\frac{\partial f}{\partial x_3}\frac{\partial F_2}{\partial p_3}-\frac{\partial f}{\partial p_3}\frac{\partial F_2}{\partial x_3}\right)=0. \tag{32}$$

The L. H. S of equation (32) is denoted by (f, F_2)

$$(f, F_2)=\sum_{r=1}^{3}\left(\frac{\partial f}{\partial x_r}\frac{\partial F_2}{\partial p_r}-\frac{\partial f}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right)=0. \tag{33}$$

Similarly consider the equations (11) and (12) and differentiate partially with respect to x_1, we get

$$\left.\begin{aligned}\frac{\partial F_1}{\partial x_1}+\frac{\partial F_1}{\partial p_1}\frac{\partial p_1}{\partial x_1}+\frac{\partial F_1}{\partial p_2}\frac{\partial p_2}{\partial x_1}+\frac{\partial F_1}{\partial p_3}\frac{\partial p_3}{\partial x_1}=0,\\ \frac{\partial F_2}{\partial x_1}+\frac{\partial F_2}{\partial p_1}\frac{\partial p_1}{\partial x_1}+\frac{\partial F_2}{\partial p_2}\frac{\partial p_2}{\partial x_1}+\frac{\partial F_2}{\partial p_3}\frac{\partial p_3}{\partial x_1}=0.\end{aligned}\right\} \tag{34}$$

Eliminating $\frac{\partial p_1}{\partial x_1}$ from (34), we have

$$\left(\frac{\partial F_1}{\partial x_1}\frac{\partial F_2}{\partial p_1}-\frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial x_1}\right)+\left(\frac{\partial F_1}{\partial p_2}\frac{\partial F_2}{\partial p_1}-\frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial p_2}\right)\frac{\partial p_2}{\partial x_1}+\left(\frac{\partial F_1}{\partial p_3}\frac{\partial F_2}{\partial p_1}-\frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial p_3}\right)\frac{\partial p_3}{\partial x_1}=0. \tag{35}$$

Differentiating equation (11) and (12) partially with respect to x_2, we get

$$\left.\begin{aligned} \frac{\partial F_1}{\partial x_2}+\frac{\partial F_1}{\partial p_1}\frac{\partial p_1}{\partial x_2}+\frac{\partial F_1}{\partial p_2}\frac{\partial p_2}{\partial x_2}+\frac{\partial F_1}{\partial p_3}\frac{\partial p_3}{\partial x_2}=0,\\ \frac{\partial F_2}{\partial x_2}+\frac{\partial F_2}{\partial p_1}\frac{\partial p_1}{\partial x_2}+\frac{\partial F_2}{\partial p_2}\frac{\partial p_2}{\partial x_2}+\frac{\partial F_2}{\partial p_3}\frac{\partial p_3}{\partial x_2}=0. \end{aligned}\right\} \tag{36}$$

Eliminating $\frac{\partial p_2}{\partial x_2}$ from (36), we have

$$\left(\frac{\partial F_1}{\partial x_2}\frac{\partial F_2}{\partial p_2}-\frac{\partial F_1}{\partial p_2}\frac{\partial F_2}{\partial x_2}\right)+\left(\frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial p_2}-\frac{\partial F_1}{\partial p_2}\frac{\partial F_2}{\partial p_1}\right)\frac{\partial p_1}{\partial x_2}+\left(\frac{\partial F_1}{\partial p_3}\frac{\partial F_2}{\partial p_2}-\frac{\partial F_1}{\partial p_2}\frac{\partial F_2}{\partial p_3}\right)\frac{\partial p_3}{\partial x_2}=0. \tag{37}$$

Again differentiating equation (11) and (12) partially with respect to x_3, we get

$$\left.\begin{aligned} \frac{\partial F_1}{\partial x_3}+\frac{\partial F_1}{\partial p_1}\frac{\partial p_1}{\partial x_3}+\frac{\partial F_1}{\partial p_2}\frac{\partial p_2}{\partial x_3}+\frac{\partial F_1}{\partial p_3}\frac{\partial p_3}{\partial x_3}=0,\\ \frac{\partial F_2}{\partial x_3}+\frac{\partial F_2}{\partial p_1}\frac{\partial p_1}{\partial x_3}+\frac{\partial F_2}{\partial p_2}\frac{\partial p_2}{\partial x_3}+\frac{\partial F_2}{\partial p_3}\frac{\partial p_3}{\partial x_3}=0. \end{aligned}\right\} \tag{38}$$

Eliminating $\frac{\partial p_3}{\partial x_3}$ from (38), we have

$$\left(\frac{\partial F_1}{\partial x_3}\frac{\partial F_2}{\partial p_3}-\frac{\partial F_1}{\partial p_3}\frac{\partial F_2}{\partial x_3}\right)+\left(\frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial p_3}-\frac{\partial F_1}{\partial p_3}\frac{\partial F_2}{\partial p_1}\right)\frac{\partial p_1}{\partial x_3}+\left(\frac{\partial F_1}{\partial p_2}\frac{\partial F_2}{\partial p_3}-\frac{\partial F_1}{\partial p_3}\frac{\partial F_2}{\partial p_2}\right)\frac{\partial p_2}{\partial x_3}=0. \tag{39}$$

Adding the equation (35), (37) and (39) and also use the relation (14), we get

$$\left(\frac{\partial F_1}{\partial x_1}\frac{\partial F_2}{\partial p_1}-\frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial x_1}\right)+\left(\frac{\partial F_1}{\partial x_2}\frac{\partial F_2}{\partial p_2}-\frac{\partial F_1}{\partial p_2}\frac{\partial F_2}{\partial x_2}\right)+\left(\frac{\partial F_1}{\partial x_3}\frac{\partial F_2}{\partial p_3}-\frac{\partial F_1}{\partial p_3}\frac{\partial F_2}{\partial x_3}\right)=0. \tag{40}$$

The L. H. S of equation (40) is denoted by (F_1, F_2)

$$(F_1,F_2)=\sum_{r=1}^{3}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r}-\frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right)=0. \tag{41}$$

Equation (25) and (33) are linear equations of first order with independent variables x_1, x_2, x_3, p_1, p_2, p_3 and dependent variables F_1, F_2 respectively. From (25) and (33) the Lagrange's auxiliary equations are,

$$\frac{dp_1}{\frac{\partial f}{\partial x_1}}=\frac{dx_1}{\frac{-\partial f}{\partial p_1}}=\frac{dp_2}{\frac{\partial f}{\partial x_2}}=\frac{dx_2}{\frac{-\partial f}{\partial p_2}}=\frac{dp_3}{\frac{\partial f}{\partial x_3}}=\frac{dx_3}{\frac{-\partial f}{\partial p_3}}. \tag{42}$$

Equation (42) are known as Jacobi's auxiliary equations. Using equation (41) we find two independent integrals $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = a_1$ and $F_2(x_1, x_2, x_3, p_1, p_2, p_3) = a_2$. If these relations satisfy (33) these are the required two additional relations (11) and (12). To find p_1, p_2, p_3 solve equation (10), (11) and (12) interms of x_1, x_2, x_3 substitute these values in (13) and then integrating the resulting equation we shall obtain a complete integral of the given equation containing the arbitrary constants of integration.

Example 2: Find a complete integral of $p_1^3 + p_2^2 + p_3 = 1$.
Solution: Given $p_1^3 + p_2^2 + p_3 = 1$, where

$$p_1 = \frac{\partial z}{\partial x_1}, \quad p_2 = \frac{\partial z}{\partial x_2}, \quad p_3 = \frac{\partial z}{\partial x_3}, \tag{43}$$

$$f(x_1, x_2, x_3, p_1, p_2, p_3) = p_1^3 + p_2^2 + p_3 - 1. \tag{44}$$

Jacobi's auxiliary equations are given by

$$\frac{dp_1}{\frac{\partial f}{\partial x_1}} = \frac{dx_1}{\frac{-\partial f}{\partial p_1}} = \frac{dp_2}{\frac{\partial f}{\partial x_2}} = \frac{dx_2}{\frac{-\partial f}{\partial p_2}} = \frac{dp_3}{\frac{\partial f}{\partial x_3}} = \frac{dx_3}{\frac{-\partial f}{\partial p_3}}, \tag{45}$$

$$\frac{dp_1}{0} = \frac{dx_1}{-3p_1^2} = \frac{dp_2}{0} = \frac{dx_2}{-2p_2} = \frac{dp_3}{0} = \frac{dx_3}{-1}. \tag{46}$$

From first and third fractions in (46)

$$dp_1 = 0 \text{ and } dp_2 = 0,$$
$$p_1 = a_1, \quad p_2 = a_2.$$

$$F_1(x_1, x_2, x_3, p_1, p_2, p_3) = p_1 = a_1, \tag{47}$$

$$F_2(x_1, x_2, x_3, p_1, p_2, p_3) = p_2 = a_2. \tag{48}$$

$$\text{Now} \quad (F_1, F_2) = \sum_{r=1}^{3} \left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r} \right) = 0,$$

$$(F_1, F_2) = \left(\frac{\partial F_1}{\partial x_1}\frac{\partial F_2}{\partial p_1} - \frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial x_1} \right) + \left(\frac{\partial F_1}{\partial x_2}\frac{\partial F_2}{\partial p_2} - \frac{\partial F_1}{\partial p_2}\frac{\partial F_2}{\partial x_2} \right) + \left(\frac{\partial F_1}{\partial x_3}\frac{\partial F_2}{\partial p_3} - \frac{\partial F_1}{\partial p_3}\frac{\partial F_2}{\partial x_3} \right) = 0.$$

Thus we have verified that for relations (47) and (48) $(F_1, F_2) = 0$. Solving equation (44), (47) and (48) for p_1, p_2, p_3, we get $\quad p_1 = a_1, \quad p_2 = a_2, \quad p_3 = 1 - a_1^3 - a_2^2,$

$$\text{Using} \quad dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3,$$

$$dz = a_1 dx_1 + a_2 dx_2 + (1 - a_1^3 - a_2^2) dx_3.$$

$$z = a_1 x_1 + a_2 x_2 + (1 - a_1^3 - a_2^2) x_3 + a_3, \tag{49}$$

where a_1, a_2, a_3 are arbitrary constants. Equation (49) is a complete integral of the given equation.

Example 3: Find a complete integral of $p_1 x_1 + p_2 x_2 - p_3^2 = 0$.
Solution:

$$\text{Given} \quad f(x_1, x_2, x_3, p_1, p_2, p_3) = p_1 x_1 + p_2 x_2 - p_3^2 = 0. \tag{50}$$

Jacobi's auxiliary equations are

$$\frac{dp_1}{p_1} = \frac{dx_1}{-x_1} = \frac{dp_2}{p_2} = \frac{dx_2}{-x_2} = \frac{dp_3}{0} = \frac{dx_3}{2p_3}. \tag{51}$$

By integrating equations from (51), we get

$$F_1(x_1, x_2, x_3, p_1, p_2, p_3) = x_1 p_1 = a_1. \tag{52}$$

$$F_2(x_1, x_2, x_3, p_1, p_2, p_3) = x_2 p_2 = a_2. \tag{53}$$

$$(F_1, F_2) = 0.$$

Thus we have verified that for relations (52) and (53) $(F_1, F_2) = 0$, solving equation (50), (52) and (53) for p_1, p_2, p_3 we get

$$p_1 = \frac{a_1}{x_1}, \quad p_2 = \frac{a_2}{x_2} \quad p_3 = (a_1 + a_2)^{\frac{1}{2}}.$$

Using

$$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3,$$

we get the complete integral as

$$z = a_1 \log x_1 + a_2 \log x_2 + (a_1 + a_2)^{\frac{1}{2}} x_3 + a_3, \tag{54}$$

where a_1, a_2, a_3 are arbitrary constants.

Example: 4 Find a complete integral of $(x_1 + x_3)(p_1 + p_2)^2 + zp_1 = 0$.
Solution:

$$\text{Given} \quad f(x_1, x_2, x_3, p_1, p_2, p_3) = (x_1 + x_3)(p_1 + p_2)^2 + zp_1. \tag{55}$$

The given equation (55) is not in the standard form, since the dependent variable z is involved. First we shall reduce it in the standard form.

$$(x_2 + x_3)\left(\frac{1}{z}\frac{\partial z}{\partial x_2} + \frac{1}{z}\frac{\partial z}{\partial x_3}\right)^2 + \frac{1}{z}\frac{\partial z}{\partial x_1} = 0. \tag{56}$$

Let $\frac{1}{z}dz = dZ \quad Z = \log z$, then we get

$$(x_2 + x_3)\left(\frac{\partial Z}{\partial x_2} + \frac{\partial Z}{\partial x_3}\right)^2 + \frac{\partial Z}{\partial x_1} = 0. \tag{57}$$

Let

$$f(x_1, x_2, x_3, p_1, p_2, p_3) = (x_2 + x_3)(p_2 + p_3)^2 + P - 1 = 0. \tag{58}$$

Jacobi's auxiliary equations are,

$$\frac{dp_1}{0} = \frac{dx_1}{-1} = \frac{dp_2}{(p_2 + p_3)^2} = \frac{dx_2}{-2(x_2 + x_3)(p_2 + p_3)} = \frac{dp_3}{(p_2 + p_3)^2} = \frac{dx_3}{-2(x_2 + x_3)(p_2 + p_3)}, \tag{59}$$

we get

$$p_1 = a_1.$$

$$p_2 = \frac{1}{2}\left[a_2 \pm \left(\frac{a_1}{x_2 + x_3}\right)^{\frac{1}{2}}\right].$$

$$p_3 = \frac{1}{2}\left[\pm\left(\frac{a_1}{x_2+x_3}\right)^{\frac{1}{2}} - a_1\right].$$

Using

$$dZ = p_1 dx_1 + p_2 dx_2 + p_3 dx_3,$$

we get complete integral

$$\log z = -a_1 x_1 + \frac{a_2}{2}(x_2 - x_3) \pm \sqrt{a_1}(x_2+x_3)^{\frac{1}{2}} + a_3.$$

where a_1, a_2, a_3 are arbitrary constants.

2.2 Hamilton-Jacobi Equation

Consider a first order nonlinear partial differential equation with $n+1$ independent variables [2, 6, 7],

$$\frac{\partial s}{\partial t} + H\left(q_i, t, \frac{\partial s}{\partial q_i}\right) = 0, \quad i = 1, 2, \ldots, n, \tag{60}$$

where q_i are the generalised coordinates, t is the time variable, H is the Hamiltonian of a dynamical system.
Using the conventional notations,
$s(q_i, t) = u(q_i, t), \quad \dfrac{\partial s}{\partial q_i} = \dfrac{\partial u}{\partial q_i} = p_i,$ and $p = s_t$.
Equation (60) in the form

$$F(q_i, t, p_i, p) = p + H(q_i, t, p_i). \tag{61}$$

The Charpit's equations of (61) in terms of parameter τ is given by

$$\frac{dq_i}{d\tau} = F_{P_i} = \frac{\partial H}{\partial p_i}, \tag{62}$$

$$\frac{dt}{d\tau} = F_P = 1, \tag{63}$$

$$\frac{du}{d\tau} = \sum_{i=1}^{n} p_i F_{p_i} + pF_p = \sum_{i=1}^{n} p_i \frac{\partial H}{\partial p_i} + p, \tag{64}$$

$$\frac{dp_i}{d\tau} = -(F_{q_i} + p_i F_u) = -\frac{\partial H}{\partial q_i}, \tag{65}$$

$$\frac{dp}{d\tau} = -(F_t + pF_u) = -\frac{\partial H}{\partial t}. \tag{66}$$

The independent variable t can be used as characteristic parameters.
From equation (63), initial condition $t(\tau) = 0$ at $\tau = 0, \quad t = \tau$.
Thus the above system of equation is written as

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \tag{67}$$

$$\frac{dp_i}{dt} = \frac{\partial H}{\partial q_i}, \tag{68}$$

$$\frac{du}{dt} = \sum_{i=1}^{n} p_i \frac{\partial H}{\partial p_i} + p, \tag{69}$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial t}. \tag{70}$$

Equation (67) and (68) constitute a set of $2n$ coupled first order ordinary differential equations i.e., Hamilton canonical equations of motion. The characteristics of the Hamilton-Jacobi equation is obtained from the solution of above equations and also they represent the generalised coordinates and generalised momenta of a dynamical system whose Hamiltonian is $H(q_i, p_i, t)$. From equation (69) and (70), we get $p = -H$ and $\Rightarrow \dfrac{du}{dt} = \sum_{i=1}^{n} p_i \dfrac{\partial H}{\partial p_i} - H$, where $q_i(t)$ and $p_i(t)$ are solved by using Hamilton system of $2n$ equations. If we substitute these solution in $\dfrac{\partial H}{\partial p_i}$ and H in (69) we get known function of t and $u(q_i, t)$ is also determined by integration. The same procedure can be used to find p by integrating equation (70). Hence, this analysis is the characteristics of the Hamilton-Jacobi equation are the solutions of the Hamilton equations. From (67) and (68) the equations of motion for any canonical function $F(q_i, p_i, t)$ can be expressed in the form

$$\frac{dF}{dt} = \{F, H\} + \frac{\partial F}{\partial t}, \tag{71}$$

where $\{F, H\}$ is called the Poisson bracket of two functions F and H. If the canonical function function F does not depend on time t, then $F_t = 0$ hence (71) becomes

$$\frac{dF}{dt} = \{F, H\}. \tag{72}$$

If F is a constant of motion then, $\{F, H\} = 0$. Generally, the Poisson bracket of two functions F and G is defined by

$$\{F, G\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} \right). \tag{73}$$

From the definition of the Poisson bracket

$$\{q_i, p_j\} = \delta_{ij}, \tag{74}$$

$$\{q_i, q_j\} = 0 = \{p_i, p_j\}, \tag{75}$$

where δ_{ij} is the Kronecker delta notation. Equation (74) and (75) represent the fundamental Poisson bracket for the canonically conjugate variable q_i and p_i. Consider the other independent variable similar to time t, i.e., replace t by q_{n+1} and equation (61) in the form

$$G(q_i, q_{n+1}, p_i, p_{n+1}) = p_{n+1} + H(q_i, t, p_i). \tag{76}$$

Hamilton equation in the form

$$\frac{dq_i}{dt} = \frac{\partial G}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial G}{\partial q_i}, \tag{77}$$

where $G = G(q_i, p_i)$ and $i = 1, 2, \ldots, n+1$. G is a system of $(2n+2)$, first-order, ordinary differential equations for unknown function q_i and p_i then the canonical transformation is of form

$$q_i = q_i(\tilde{q}_i, \tilde{p}_i), \quad p_i = p_i(\tilde{q}_i, \tilde{p}_i), \tag{78}$$

where $i = 1, 2, \dots, n+1$ is called canonical if the function $\tilde{G}(\tilde{q_i}, \tilde{p_i})$ exists then (78) transform into

$$\frac{d\tilde{q_i}}{dt} = \frac{\partial \tilde{G}}{\partial p_i}, \quad \frac{d\tilde{p_i}}{dt} = -\frac{\partial \tilde{G}}{\partial q_i}. \tag{79}$$

A theorem of Poincare states that,

$$J_1 = \iint_s \sum_i dq_i dp_i. \tag{80}$$

These is the other mathematical expressions invariant under the canonical transformation, where S indicates the integral is to be evaluated over any arbitrary two-dimensional surface formed by coordinates $q_1, q_2, \dots, q_n$ and $p_1, p_2, \dots, p_n$. Assume u and v are two parameters appropriate to the surface s then $q_i = q_i(u, v)$ and $p_i = p_i(u, v)$ on this surface. The element $dudv$ transform from the elementary area $dq_i dp_i$ according to the relation

$$dq_i dp_i = \frac{\partial(q_i, p_i)}{\partial(u, v)} dudv, \tag{81}$$

where the Jacobian determinant is given by

$$\frac{\partial(q_i, p_i)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial p_i}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial p_i}{\partial v} \end{vmatrix} \neq 0. \tag{82}$$

The integral J_1 has the same value for all canonical coordinates, that is,

$$\iint_s \left(\sum_i dq_i dp_i \right) = \iint_s \left(\sum_k d\tilde{q_k} d\tilde{p_k} \right), \tag{83}$$

equation (83) can also be written as

$$\iint_s \sum_i \frac{\partial(q_i, p_i)}{\partial(u, v)} dudv = \iint_s \sum_k \frac{\partial(\tilde{q_k}, \tilde{p_k})}{\partial(u, v)} dudv. \tag{84}$$

Since the region of integration is arbitrary, in equation (84) if the two integrands are equal then two integrals are equal,

$$\sum_i \frac{\partial(q_i, p_i)}{\partial(u, v)} = \sum_k \frac{\partial(\tilde{q_k}, \tilde{p_k})}{\partial(u, v)}. \tag{85}$$

i.e., the sum of the Jacobian determinant is invariant. Equation (85) can also be written as

$$\sum_i \left(\frac{\partial q_i}{\partial u} \frac{\partial p_i}{\partial v} - \frac{\partial p_i}{\partial u} \frac{\partial q_i}{\partial v} \right) = \sum_k \left(\frac{\partial \tilde{q_k}}{\partial u} \frac{\partial \tilde{p_k}}{\partial v} - \frac{\partial \tilde{p_k}}{\partial u} \frac{\partial \tilde{q_k}}{\partial v} \right). \tag{86}$$

Equation (86) is of the form Lagrange bracket of two independent variable u and v. Generally,if u and v are two independent variables, and F_k and G_k, $k = 1, 2, \dots, n$ are a set of functions then the Lagrange bracket is defined by

$$(u, v) = \sum_i \left(\frac{\partial F_k}{\partial u} \frac{\partial G_k}{\partial v} - \frac{\partial G_k}{\partial u} \frac{\partial F_k}{\partial v} \right). \tag{87}$$

Consider the Lagrange bracket of the generalized coordinates

$$(q_i, q_j) = \sum_k \left(\frac{\partial q_k}{\partial q_i} \frac{\partial p_k}{\partial q_i} - \frac{\partial q_k}{\partial q_j} \frac{\partial p_k}{\partial q_i} \right) = 0. \tag{88}$$

Since q's and p's are independent coordinates, hence $\frac{\partial p_k}{\partial q_i} = 0$ and $\frac{\partial p_k}{\partial q_j} = 0$. We get $(q_i, q_j) = 0$, similarly $(p_i, p_j) = 0$.

$$(q_i, p_j) = \sum_k \left(\frac{\partial q_k}{\partial q_i} \frac{\partial p_k}{\partial p_j} - \frac{\partial q_k}{\partial p_j} \frac{\partial p_k}{\partial q_i} \right). \tag{89}$$

Here p's and q's are independent, so the second term in the (89) vanishes, but the first term is not zero because

$$\frac{\partial q_k}{\partial q_i} = \delta_{ki}, \quad \frac{\partial p_k}{\partial p_j} = \delta_{kj}, \tag{90}$$

hence $(q_i, p_j) = \delta_{ij}$. Thus equation (89) and (88) represent the Lagrange bracket of canonical variables.
Consider a conservative dynamical system

$$S = u - Et,$$

where u is independent of time t, E is an arbitrary constant, and S is a solution of the Hamilton-Jacobi equation (60). If u satisfies the time independent Hamilton-Jacobi equation in the form,

$$H\left(q_i, \frac{\partial u}{\partial q_i} \right) = E. \tag{91}$$

Example 5: Let a particle of mass m moving in a plane with potential energy $V(x, y)$ [7]. The Lagrangian is

$$L = \frac{1}{2} m(\dot{x^2} + \dot{y^2}) - V(x, y), \tag{92}$$

The generalized momenta are

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}.$$

$$\dot{x} = \frac{p_x}{m}, \qquad \dot{y} = \frac{p_y}{m}.$$

The Hamiltonian is given by

$$H = T + V = \frac{1}{2} m(\dot{x^2} + \dot{y^2}) - V(x, y), \tag{93}$$

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + V(x, y). \tag{94}$$

We have $p_i = \frac{\partial s}{\partial q_i} = \frac{\partial u}{\partial q_i}, \quad i = 1, 2$. We may use Cartesian coordinates as the generalised coordinates, that is, $q_1 = x$ $q_2 = y$,

$$\frac{1}{2} m \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] - V(x, y) = E. \tag{95}$$

Equivalently, $u_x^2 + u_y^2 = f(x,y)$, where $f(x,y) = 2m(E - V)$.

Example 6: Solve the Hamilton-Jacobi equation for one dimensional harmonic oscillator [6]. Consider the Hamilton is

$$H = \frac{p^2}{2m} + \frac{1}{2}kq^2, \tag{96}$$

where m is mass of the particle and k is the spring constant. We have $s = s(q,p,t)$ and $p = \dfrac{\partial s}{\partial q}$. Using Hamilton-Jacobi equation, we get

$$\frac{1}{2m}\left(\frac{\partial s}{\partial q}\right)^2 + \frac{1}{2}kq^2 + \frac{\partial s}{\partial t} = 0. \tag{97}$$

Then the function F is given by

$$F(t,q,s,s_t,s_q) = \frac{1}{2m}s_q^2 + \frac{1}{2}kq^2 + s_t. \tag{98}$$

The Charpit's subsidiary equations are,

$$\frac{dt}{1} = \frac{dq}{\frac{s_q}{m}} = \frac{ds}{s_t + \frac{s_q^2}{m}} = \frac{ds_t}{0} = \frac{ds_q}{-kq}. \tag{99}$$

Consider first and last term in equation (99) we get

$$\frac{s_q^2}{2m} + \frac{1}{2}kq^2 = a, \tag{100}$$

where a is the constant of integration. From equation (99), we get

$$s_t = -a, \quad s_q = \pm\sqrt{mk}\left[\frac{2a}{k} - q^2\right]^{\frac{1}{2}}, \tag{101}$$

$$ds = s_t dt + s_q dq, \tag{102}$$

$$s = -at \pm\sqrt{mk}\int\left[\frac{2a}{k} - q^2\right]^{\frac{1}{2}} dq + b. \tag{103}$$

Consider $\beta = \dfrac{\partial s}{\partial a}$.

$$\beta = -t \pm\sqrt{\frac{m}{k}}\int\left[\frac{dq}{\frac{2a}{k} - q^2}\right]^{\frac{1}{2}}. \tag{104}$$

Integrating and rearranging the terms, we get

$$\beta + t = \pm\sqrt{\frac{m}{k}}cos^{-1}\left(\frac{q}{b}\right),$$

$$q = \sqrt{\frac{2a}{k}}\cos\omega(\beta + t), \tag{105}$$

where $\omega = \sqrt{\frac{k}{m}}$, ω is angular frequency of oscillation and a is the energy of particle.

2.3 Hamilton-Jacobi Theory

Hamilton-Jacobi shows that "All the solutions of the ordinary differential equations can be obtained with the help of a partial differential equation" [1, 6, 7].

$$\text{Let} \quad \frac{dx_\alpha}{dt} = H_{p_\alpha}, \quad \frac{dp_\alpha}{dt} = -H_{x_\alpha}, \tag{106}$$

where $H = H(t; x_1, x_2, \ldots, x_m; p_1, p_2, \ldots, p_m)$. Equation (106) is referred to as Hamilton's canonical equations of a conservative mechanical system with m degrees of freedom. The independent variable t is time, H is the Hamiltonian of the system and $x_1, x_2, \ldots, x_m$ are generalized coordinates, $p_1, p_2, \ldots, p_m$ are generalized momenta.
Consider the partial differential equation

$$q + H(t; x_1, x_2, \ldots, x_m; p_1, p_2, \ldots, p_m) = 0, \tag{107}$$

where the dependent variable does not appear explicitly. Equation (107) is called the Hamilton-Jacobi equation. The Charpit's equations of (107) are

$$\frac{dx_\alpha}{d\sigma} = H_{P_\alpha}, \tag{108}$$

$$\frac{dt}{d\sigma} = 1, \tag{109}$$

$$\frac{du}{d\sigma} = q + p_\alpha H_{P_\alpha}, \tag{110}$$

$$\frac{dp}{d\sigma} = -H_{x_\alpha}, \tag{111}$$

$$\frac{dq}{d\sigma} = -H_t. \tag{112}$$

From equation (109) in the Charpit's equations we can replace the variable σ by t.

$$\frac{dx_\alpha}{dt} = H_{P_\alpha}, \tag{113}$$

$$\frac{du}{dt} = q + p_\alpha H_{P_\alpha}, \tag{114}$$

$$\frac{dp}{dt} = -H_{x_\alpha}, \tag{115}$$

$$\frac{dq}{dt} = -H_t. \tag{116}$$

The equations (113) and (114) are nothing but the Hamilton's canonical equations of (106). In the function H, u and q does not appear. Here x_α and p_α have been determined from Hamilton's canonical equations and q and u can be determined from equation (107) and (114). The totality of all Monge strips of equation (107) form $(2m + 1)$ parameter family of strips. Since u does not appear explicitly in the equations the strips depend only on $2m$ parameter. Therefore the set of all Monge strips depend only on $2m$parameter solutions of the Hamilton's canonical equation(106). We have to show that every solutions of equation (106) leads to a Monge strip. The function $q + H$ remains constant along every solutions of the equations. Therefore $x_\alpha = x_\alpha(t), \quad p_\alpha = p_\alpha(t)$, at $t = 0,\ q(0) + H(0, x_\alpha(0), p_\alpha(0)) = 0$, thus solving equation (116) and (114) with an arbitrary initial value of u this gives a (2m+1) parameter

Monge strips of the Hamilton-Jacobi equations. If u be any solution of equation (107) then $u + b$ is also the solution of equation (107), where b is constant.
Let $u = \phi(x_1, x_2, \ldots, x_m; t; a_1, a_2, \ldots, a_m)$ be the solutions of equation (109) depending on m, depending on m with arbitrary parameters $a_1, a_2, \ldots a_m$,

$$\Delta = |\phi_{x_\alpha\, a_\beta}| \neq 0. \tag{117}$$

Then

$$u = \phi(x_1, x_2, \ldots, x_m; t; a_1, a_2 \ldots a_m) + b, \tag{118}$$

depending on $m + 1$ parameters a_x, b is complete integral of equation (107). The general solution of equation (107) is obtained by assuming $b = b(a_1, a_2, \ldots, a_m)$ as an arbitrary function of a_α and the solution of equation (118) is obtained by eliminating $a_1, a_2 \ldots a_m$ from equation (118) and then differentiating equation (118) with respect to a_α gives the m relations.

$$\phi_{x\alpha} + b_\alpha = 0, \tag{119}$$

where $b_\alpha = \dfrac{\partial b}{\partial a_\alpha}(a_1, a_2, \ldots a_m)$. If a_α, b, b_α are arbitrary constants then $(m + 1)$ relations of equation (119) and (118) represents the $(2m+1)$ parameters family of Monge curve of equation (107) in (x_t, t, u) space. The variation of p_α and q along these curves is given by

$$p_\alpha = \phi_{x\alpha}(x_1, x_2, \ldots x_m; t; a_1, a_2, \ldots a_m). \tag{120}$$

That is for constant values of a_α, b, b_α the equations (107), (118), (119) and (120) give the $(2m+1)$ parameter family of Monge strips of equation (107) and u, q, b do not appear in equation (119) and (120). Therefore these equations together represent the $2m$ parameter family of solutions of the Hamilton's canonical equations (106).

3 Conclusion

In this paper we have solved nonlinear partial and ordinary differential equations of higher dimensions by using Jacobi theory which is a very strong method. The other methods used are based by Charpits method for higher dimensions. These methods are very useful to solve highly nonlinear ODE and PDE arising in many applications in fluid dynamics and mechanical engineering.

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Solution of Ordinary Differential Equations using 3-scale Haar Wavelets

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Abstract: Ordinary differential equations arise in the modelling of many physical phenomena. In this paper, we use 3-scale Haar wavelet method for the numerical solution of initial value problems and boundary value problems. The basic idea of Haar wavelet collocation method is to convert the ordinary differential equation into a system of algebraic equations that involve a finite number of variables. The numerical results are compared with the exact solution to prove the accuracy of the Haar wavelet method.

Keywords: Ordinary differential equations, Initial value problem, Boundary value problem, 3-scale Haar wavelets, Collocation points.

1 Introduction

An Ordinary Differential Equation (ODE) is a differential equation containing one or more function of one independent variable and its derivatives. Ordinary differential equations are applicable in Newton's law of cooling, electrical circuits, modeling free mechanical oscillation, modeling forced mechanical oscillations, computer exercise or activity etc. Initial value problems (IVP) have all the conditions specified at the same value of the independent variable in the equation. Boundary value problems (BVP) have conditions at the extremes of the independent variable in the equation.

The first known connection to modern wavelets dates back to the French mathematician Jean Baptiste Joseph Fourier. In 1807, Fourier's efforts with frequency analysis lead to what we now know as Fourier Analysis. His work was based on the fact that periodic functions can be represented in terms of trigonometric series. Fourier developed the idea of Fourier series so that it can be used as a practical tool for determining the Fourier series solution of partial differential equations under prescribed boundary conditions. The well-known Fourier integral theorem was formulated in order to obtain a representation for a non-periodic function. The Fourier Transform originated from the Fourier integral theorem and it transforms a function which depends on time into a new function which depends on frequency. The incredible triumph of the Fourier transform analysis is due to the fact that the function can be reconstructed by the Fourier inversion formula.

Fourier series and transformations are very popular and powerful tools in mathematics and many other disciplines. The power and flexibility of Fourier analysis have facilitated an incredibly diverse range of applications to modern mathematics, science and engineering. A non-stationary signal like speech can be represented in the frequency domain by Fourier transform as though it has been synthesized by many frequency components of infinite duration and can be recovered by inverse Fourier transform. However, the Fourier transform does not explicitly indicate or localize when a particular frequency characteristic occurred, as it assumes that the same spectrum occurs for all the time [1, 2].

Multiresolution analysis and wavelet transform will help to localize the frequency analysis. In the year 1909, Alfred Haar, a Hungarian mathematician introduced Haar function which were later known as Haar wavelets. His contribution to wavelets is very evident. There is an entire wavelet family named after him. The Haar wavelet is a sequence of rescaled "square-shaped" functions which together form a wavelet family or basis. They consist of piecewise constant functions and are therefore the simplest orthonormal wavelets with a compact support. An advantage of these wavelets is the possibility to integrate them analytically for arbitrary times. They are conceptually simple, fast, memory efficient and exactly reversible [3].

Wavelet theory is the result of a multidisciplinary effort that brought together mathematicians, physicists and engineers. The connection has created a flow of ideas that goes well beyond the construction of new bases or transforms. Wavelet theory has become an effective tool for the development of pure and applied mathematics. Wavelets are well-suited for approximating data with sharp discontinuities. Wavelet representation is more accurate and useful in data compression, noise removal, pattern classification and fast scientific computation. Wavelets are mathematical functions that decompose data into different frequency components and then each component is studied with a resolution matched to its scale [4].

In recent years, the wavelet approach for the solution of differential and integral equations has become very popular. Multiresolution analysis of wavelets capture local features efficiently as such enables to detect singularities, shocks, irregular structure and transient phenomena exhibited by the analyzed equations. Chen and Hsiao [5] recommended to expand into the Haar series the highest order derivatives appearing in the differential equation. This idea has been very prolific and it is being abundantly applies for the solution of differential equations. The wavelet coefficients appearing in the Haar series are calculated either using Collocation method or Galerkin method.

Over the recent decades, wavelets by and large have picked up a respectable status because of their applications in different disciplines and in that capacity have many success stories. Prominent effects of their studies are in the fields of signal processing, computer vision, seismology, turbulence, computer graphics, image processing, structures of the galaxies in the universe, digital communication, pattern recognition, approximation theory, quantum optics, biomedical engineering, sampling theory, matrix theory, operator theory, differential equations, integral equations, numerical analysis, statistics, tomography, and so on. A standout amongst the best utilizations of wavelets has been in image processing. The Federal Bureau of Investigation (FBI) has build up a wavelet based algorithm for fingerprint compression. Wavelets have the capability to designate functions at different levels of resolution, which permits building up a chain of approximate solutions of equations. Compactly supported wavelets are localized in space, wherein solutions can be refined in regions of sharp variations/transients without going for new grid generation, which is the common strategy in classical numerical schemes.

Chang and Piau [6], Fazali-i-Haq and Ali [7] have solved higher order boundary value and eigenvalue problems using Haar wavelets. Lepik [8] applied the Haar wavelet method along with the segmentation technique to solve differential equations. Lepik [9] used Haar wavelets for solving higher order differential equations. Lepik [10] solved differential equations with the aid of non-uniform Haar wavelets. Dhawan et al. [11] used wavelet based numerical scheme to solve differential equations. Hsiao and Wu [12] used Haar wavelets to solve time-varying functional differential equations. Khalid et al. [13] solved Airy differential equation using Haar wavelets. Shi and Cao [14] applied Haar wavelets to solve eigenvalue problems of higher order differential equations. Mohammadi et al. [15] and Yousefi [16] used Legendre wavelets to solve singular ordinary differential equations and Lane-Emden type differential equations respectively. Kesava et al. [17] solved ordinary differential equations and system of ordinary differential equations, both initial value problems and boundary value problems, using 2-scale Haar wavelets.

In this chapter, we attempt to solve ordinary differential equations, both initial value problem and boundary value problem using 3-scale Haar wavelets.

2 Haar Wavelet

The Haar wavelet family [18] for $x \in, [0, 1]$ is defined as follows,

$$h_i(x) = \begin{cases} \psi_i^1(x) & \text{for even } i, \\ \\ \psi_i^2(x) & \text{for odd } i, \end{cases} \tag{1}$$

where

$$\psi_i^1(x) = \frac{1}{\sqrt{2}} \begin{cases} -1 & \text{for } \xi_1 \leq x \leq \xi_2, \\ \\ 2 & \text{for } \xi_2 \leq x \leq \xi_3, \\ \\ -1 & \text{for } \xi_3 \leq x \leq \xi_4, \\ \\ 0 & \text{elsewhere}, \end{cases} \tag{2}$$

$$\psi_i^2(x) = \sqrt{\frac{3}{2}} \begin{cases} 1 & \text{for } \xi_1 \leq x \leq \xi_2, \\ \\ 0 & \text{for } \xi_2 \leq x \leq \xi_3, \\ \\ -1 & \text{for } \xi_3 \leq x \leq \xi_4, \\ \\ 0 & \text{elsewhere}, \end{cases} \tag{3}$$

$$\xi_1 = \frac{k}{m}, \quad \xi_2 = \frac{k + \frac{1}{3}}{m}, \quad \xi_3 = \frac{k + \frac{2}{3}}{m}, \quad \xi_4 = \frac{k+1}{m}. \tag{4}$$

In the above definition $m = 3^j$, $j = 0, 1, \ldots, J$ indicates the level of the wavelet; $k = 0, 1, \ldots m - 1$ is the translation parameter. J is the maximum level of resolution. For index $i = 1$, $h_1(x)$ is assumed to be the scaling function which is defined as follows.

$$h_1(x) = \begin{cases} 1 & \text{for } x \in [0, 1) \\ \\ 0 & \text{elsewhere} \end{cases} \tag{5}$$

For index $i > 1$, even and odd index are calculated from the formula $i = m + 2k + 1$ and $i = m + 2k + 2$ respectively.

In order solve differential equations of any order, we need the following integrals.

$$p_i(x) = \int_0^x h_i(x)dx = \begin{cases} \theta_i^1(x) = \int_0^x \psi_i^1(x)dx & \text{for even } i, \\ \theta_i^2(x) = \int_0^x \psi_i^2(x)dx & \text{for odd } i, \end{cases} \tag{6}$$

where

$$\theta_i^1(x) = \frac{1}{\sqrt{2}} \begin{cases} \xi_1 - x & \text{for } \xi_1 \le x \le \xi_2, \\ 2x - 3\xi_2 + \xi_1 & \text{for } \xi_2 \le x \le \xi_3, \\ \xi_1 - 3\xi_2 + 3\xi_3 - x & \text{for } \xi_3 \le x \le \xi_4, \\ 0 & \text{elsewhere}, \end{cases} \tag{7}$$

$$\theta_i^2(x) = \sqrt{\frac{3}{2}} \begin{cases} x - \xi_1 & \text{for } \xi_1 \le x \le \xi_2, \\ \xi_2 - \xi_1 & \text{for } \xi_2 \le x \le \xi_3, \\ \xi_3 + \xi_2 - \xi_{-}x & \text{for } \xi_3 \le x \le \xi_4, \\ 0 & \text{elsewhere}. \end{cases} \tag{8}$$

$$q_i(x) = \int_0^x p_i(x)dx = \begin{cases} \zeta^1(x) = \int_0^x \theta_i^1(x)dx & \text{for even } i, \\ \zeta^2(x) = \int_0^x \theta_i^2(x)dx & \text{for odd } i, \end{cases} \tag{9}$$

where

$$\zeta_i^1(x) = \frac{1}{2\sqrt{2}} \begin{cases} -(\xi_1 - x)^2 & \text{for} \xi_1 \le x \le \xi_2, \\ 2(x - 2\xi_2 + \xi_1)(x - \xi_2) - (\xi_1 - \xi_2)^2 & \text{for } \xi_2 \le x \le \xi_3, \\ (3\xi_3 - 2\xi_2 - x)(x - \xi_3) - (\xi_1 - \xi_2)^2 & \text{for } \xi_3 \le x \le \xi_4, \\ 0 & \text{elsewhere}, \end{cases} \tag{10}$$

$$\zeta_i^2(x) = \frac{1}{2}\sqrt{\frac{3}{2}} \begin{cases} (x - \xi_1)^2 & \text{for } \xi_1 \le x \le \xi_2, \\ (\xi_2 - \xi_1)(2x - \xi_2 - \xi_1) & \text{for } \xi_2 \le x \le \xi_3, \\ (x - \xi_3)(\xi_3 + 2\xi_2 - 2\xi_1 - x) \\ \quad + (\xi_2 - \xi_1)(2\xi_3 - \xi_2 - \xi_1) & \text{for } \xi_3 \le x \le \xi_4, \\ (\xi_4 - \xi_3)(\xi_3 + 2\xi_2 - 2\xi_1 - \xi_4) \\ \quad + (\xi_2 - \xi_1)(2\xi_3 - \xi_2 - \xi_1) & \text{for } \xi_4 \le x \le 1, \\ 0 & \text{elsewhere}. \end{cases} \tag{11}$$

2.1 Function approximation

Any function $f(x)$ which is square integrable on $[0,1)$ can be expressed as an infinite sum of Haar wavelets as

$$f(x) = \sum_{i=1}^{\infty} a_i h_i(x), \tag{12}$$

where

$$a_i = \int_0^1 f(x) h_i(x) dx. \tag{13}$$

If $f(x)$ is approximated as piecewise constant in each subinterval, then equation (12) will be terminated at finite terms, i.e.

$$f(x) = \sum_{i=1}^{3M} a_i h_i(x), \tag{14}$$

or,

$$f(x) = a_1 h_1(x) + \sum_{\substack{i=2 \\ \text{even } i}}^{3M} a_i \psi_i^1(x) + \sum_{\substack{i=3 \\ \text{odd } i}}^{3M} a_i \psi_i^2(x), \tag{15}$$

where the wavelet coefficients $a_i,\ i = 1, 2, \ldots, 3M$ are to be determined.

3 Method of Solution

In this section, the description of the Haar wavelet collocation method to solve linear ordinary differential equations of second order is outlined for both initial value problems and boundary value problems.

3.1 Initial Value Problems

Consider the linear ordinary differential equation of second order,

$$c(x)y''(x) + d(x)y'(x) + f(x)y(x) = g(x), \tag{16}$$

with initial conditions,

$$y(0) = \alpha, y'(0) = \beta, \tag{17}$$

where $c(x), d(x), f(x), g(x)$ are known functions and α, β are constants.

Let the Haar wavelet solution be in the form,

$$y''(x) = \sum_{i=1}^{3M} a_i h_i(x). \tag{18}$$

Integrating equation (18) with respect x from 0 to x and using equation (17) gives

$$y'(x) = \beta + \sum_{i=1}^{3M} a_i p_i(x). \tag{19}$$

Integrating equation (19) with respect x from 0 to x and using equation (17) leads to

$$y(x) = \alpha + \beta x + \sum_{i=1}^{3M} a_i q_i(x). \tag{20}$$

Substituting equations (18), (19) and (20) in equation (16), we obtain

$$\sum_{i=1}^{3M} a_i[c(x)h_i(x) + d(x)p_i(x) + f(x)q_i(x)] = g(x) - \alpha f(x) - \beta[xf(x) + d(x)]. \tag{21}$$

The wavelet collocation points are defined as,

$$x_l = \frac{l - 0.5}{3M}, l = 1, 2, \ldots, 3M \tag{22}$$

Taking the collocation points $x \to x_l$ in equations (21) and (20), we get

$$\sum_{i=1}^{3M} a_i[c(x_l)h_i(x_l) + d(x_l)p_i(x_l) + f(x_l)q_i(x_l)] = g(x_l) - \alpha f(x_l) - \beta[x_l f(x_l) + d(x_l)], \tag{23}$$

$$y(x_l) = \alpha + \beta(x_l) + \sum_{i=1}^{3M} a_i q_i(x). \tag{24}$$

3.2 Boundary Value Problems

Consider the linear ordinary differential equation of second order,

$$c(x)y''(x) + d(x)y'(x) + f(x)y(x) = g(x), \tag{25}$$

with boundary conditions,

$$y(0) = \alpha, y(1) = \beta, \tag{26}$$

where $c(x), d(x), f(x), g(x)$ are known functions and α, β are constants.

Let the Haar wavelet solution be in the form

$$y''(x) = \sum_{i=1}^{3M} a_i h_i(x). \tag{27}$$

Integrating equation (27) with respect x from 0 to x gives

$$y'(x) = y'(0) + \sum_{i=1}^{3M} a_i p_i(x)dx. \tag{28}$$

Integrating equation (28) with respect x from 0 to x and using equation (26) leads to

$$y(x) = \alpha + xy'(0) + \sum_{i=1}^{3M} a_i q_i(x)dx. \tag{29}$$

Putting $x = 1$ in equation (29) and using equation (26), we obtain

$$y'(0) = \beta - \alpha - \sum_{i=1}^{3M} a_i q_i(1). \tag{30}$$

Substituting equation (30) in equations (28) and (29), we get

$$y'(x) = \beta - \alpha + \sum_{i=1}^{3M} a_i [p_i(x) - q_i(1)], \tag{31}$$

$$y(x) = \alpha(1 - x) + \beta x + \sum_{i=1}^{3M} a_i [q_i(x) - x q_i(1)]. \tag{32}$$

Substituting equations (27), (31) and (32) in equation (25) gives

$$\begin{aligned} \sum_{i=1}^{3M} a_i [c(x)h_i(x) + d(x)p_i(x) + f(x)q_i(x) - dx + xf(x)q_i(1)] = g(x)+ \\ \alpha[dx - (1 - x)f(x)] - \beta[d(x) + xf(x)]. \end{aligned} \tag{33}$$

Taking the collocation points $x \to x_l$ in equations (33) and (32) leads to

$$\begin{aligned} \sum_{i=1}^{3M} a_i [c(x_l)h_i(x_l) + d(x_l)p_i(x_l) + f(x_l)q_i(x_l) - d(x_l + x_l f_(x_l)q_i(1)] = g(x_l)+ \\ \alpha[d(x_l) - (1 - x_l)f(x_l)] - \beta[d(x_l) + x_l f(x_l)], \end{aligned} \tag{34}$$

$$y(x_l) = \alpha(1 - x_l) + \beta x_l + \sum_{i=1}^{3M} a_i [q_i(x_l) - x_l q_i(1)]. \tag{35}$$

ERROR ESTIMATE: We define the error estimate by

$$\sigma = \frac{1}{3M} \| y(x) - y_{ex}(x) \| \tag{36}$$

where $y_{ex}(x)$ is the exact solution.

4 Examples and Discussions

In this section, five examples each of initial value problems and boundary value problems for linear ordinary differential equations of second order are discussed. The entire computational work is done using Scilab.

Example 1:

$$y'' - 3y' + 2y = 6e^{-x}, \tag{37}$$

with initial conditions

$$y(0) = y'(0) = 2. \tag{38}$$

The exact solution is

$$y(x) = 2e^{2x} - e^{x} + e^{-x}. \tag{39}$$

Solving equations (21) and (20), we obtain

$$\sum_{i=1}^{3M} a_i[h_i(x) - 3p_i(x) + 2q_i(x)] = 2 - 4x + 6e^{-x}, \tag{40}$$

$$y(x) = 2 + 2x + \sum_{i=1}^{3M} a_i q_i(x). \tag{41}$$

The HWCM solution of this example with $J = 4$ is given in Table 1 The results are compared with the exact solution and are found to be in good agreement. Figures 1 shows the comparison of the HWCM solution with the exact solution. The error estimates obtained for different J are given in Table 2.

Example 2:

$$y'' - 2y' + y = xe^x - x, \tag{42}$$

with initial conditions

$$y(0) = y'(0) = 0. \tag{43}$$

The exact solution is

$$y(x) = \frac{1}{6}x^3e^x - xe^x + 2e^x - x - 2. \tag{44}$$

Solving equations (21) and (20), we obtain

$$\sum_{i=1}^{3M} a_i[h_i(x) - 2p_i(x) + q_i(x)] = xe^x - x, \tag{45}$$

$$y(x) = \sum_{i=1}^{3M} a_i q_i(x). \tag{46}$$

The HWCM solution of this example with $J = 4$ is given in Table 3. The results are compared with the exact solution and are found to be in good agreement. Figures 2 shows the comparison of the HWCM solution with the exact solution. The error estimates obtained for different J are given in Table 4.

Example 3:

$$y'' - 3y' + 2y = 4\sin(x), \tag{47}$$

with initial conditions

$$y(0) = 0, y'(0) = -1. \tag{48}$$

The exact solution is

$$y(x) = \frac{2}{5}(3\cos(x)) + \sin(x) - e^x - \frac{1}{5}e^2x. \tag{49}$$

Solving equations (21) and (20), we obtain

$$\sum_{i=1}^{3M} a_i[h_i(x) - 3p_i(x) + 2q_i(x)] = 4\sin(x) - 3 - 2x, \tag{50}$$

$$y(x) = -x + \sum_{i=1}^{3M} a_i q_i(x). \tag{51}$$

The HWCM solution of this example with $J = 4$ is given in Table 5. The results are compared with the exact solution and are found to be in good agreement. Figures 3 shows the comparison of the HWCM solution with the exact solution. The error estimates obtained for different J are given in Table 6.

Example 4:

$$y'' + 16y = 1, \tag{52}$$

with initial conditions

$$y(0) = 1, y'(0) = 2. \tag{53}$$

The exact solution is

$$y(x) = \frac{15}{16}(\cos(4x)) + \frac{1}{2}\sin(4x) + \frac{1}{16}. \tag{54}$$

Solving equations (21) and (20), we obtain

$$\sum_{i=1}^{3M} a_i[h_i(x) + 16q_i(x)] = -15 - 32x, \tag{55}$$

$$y(x) = 1 + 2x + \sum_{i=1}^{3M} a_i q_i(x). \tag{56}$$

The HWCM solution of this example with $J = 4$ is given in Table 7. The results are compared with the exact solution and are found to be in good agreement. Figures 4 shows the comparison of the HWCM solution with the exact solution. The error estimates obtained for different J are given in Table 8.

Example 5:

$$y'' + 4y = 4\cos(2x), \tag{57}$$

with initial conditions

$$y(0) = 0, y'(0) = 2. \tag{58}$$

The exact solution is

$$y(x) = \sin(x) + x\sin(2x) \tag{59}$$

Solving equations (21) and (20), we obtain

$$\sum_{i=1}^{3M} a_i[h_i(x) + 4q_i(x)] = 4\cos 2(x) - 8x, \tag{60}$$

$$y(x) = 2x + \sum_{i=1}^{3M} a_i q_i(x). \tag{61}$$

The HWCM solution of this example with $J = 4$ is given in Table 9. The results are compared with the exact solution and are found to be in good agreement. Figures 5 shows the comparison of the HWCM solution with the exact solution. The error estimates obtained for different J are given in Table 10.

Example 6:

$$y'' + y = 1, \tag{62}$$

with boundary conditions

$$y(0) = 0, y(1) = 1. \tag{63}$$

The exact solution is

$$y(x) = -\cos(x) + \cot(1)\sin(x) + 1. \tag{64}$$

Solving equations (33) and (32), we obtain

$$\sum_{i=1}^{3M} a_i[h_i(x) + q_i(x) - xq_i(1)] = 1 - x, \tag{65}$$

$$y(x) = \sum_{i=1}^{3M} a_i[q_i(x) - q_i(1)]. \tag{66}$$

The HWCM solution of this example with $J = 4$ is given in Table 11. The results are compared with the exact solution and are found to be in good agreement. Figures 6 shows the comparison of the HWCM solution with the exact solution. The error estimates obtained for different J are given in Table 12.

Example 7:

$$y'' + y + x = 0, \tag{67}$$

with boundary conditions

$$y(0) = 0, y(1) = 0. \tag{68}$$

The exact solution is

$$y(x) = \frac{\sin(x)}{\sin(1)} - x. \tag{69}$$

Solving equations (33) and (32), we obtain

$$\sum_{i=1}^{3M} a_i[h_i(x) + q_i(x) - xq_i(1)] = -x, \tag{70}$$

$$y(x) = \sum_{i=1}^{3M} a_i[q_i(x) - q_i(1)]. \tag{71}$$

The HWCM solution of this example with $J = 4$ is given in Table 13. The results are compared with the exact solution and are found to be in good agreement. Figures 7 shows the comparison of the HWCM solution with the exact solution. The error estimates obtained for different J are given in Table 14.

Example 8.:

$$y'' - y = 0, \tag{72}$$

with boundary conditions

$$y(0) = 0, y(1) = 1. \tag{73}$$

The exact solution is

$$y(x) = \frac{e^{1-x}(e^{2x} - 1)}{e^2 - 1}. \tag{74}$$

Solving equations (33) and (32), we obtain

$$\sum_{i=1}^{3M} a_i[h_i(x) + q_i(x) + xq_i(1)] = x, \tag{75}$$

$$y(x) = \sum_{i=1}^{3M} a_i[q_i(x) - xq_i(1)] + x. \tag{76}$$

The HWCM solution of this example with $J = 4$ is given in Table 15. The results are compared with the exact solution and are found to be in good agreement. Figures 8 shows the comparison of the HWCM solution with the exact solution. The error estimates obtained for different J are given in Table 16.

Example 9:

$$y'' - y = 0, \tag{77}$$

with boundary conditions

$$y(0) = 0, y(1) = 0. \tag{78}$$

The exact solution is

$$y(x) = \frac{e^2(e^{2x} - e^{-2x}}{e^4 - 1)}. \tag{79}$$

Solving equations (33) and (32), we obtain

$$\sum_{i=1}^{3M} a_i(x)[h_i(x) - 4q_i(x) + 4xq_i(1)] = 4x, \tag{80}$$

$$y(x) = 2x + \sum_{i=1}^{3M} a_i[q_i(x) - xq_i(1). \tag{81}$$

The HWCM solution of this example with $J = 4$ is given in Table 17. The results are compared with the exact solution and are found to be in good agreement. Figures 9 shows the comparison of the HWCM solution with the exact solution. The error estimates obtained for different J are given in Table 18.

Example 10:

$$y'' + \cos(\pi(x) = 0, \tag{82}$$

with boundary conditions

$$y(0) = 0, y(1) = 0. \tag{83}$$

The exact solution is

$$y(x) = \frac{1}{\pi^2}[\cos \pi(x) + 2x - 1]. \tag{84}$$

Solving equations (33) and (32), we obtain

$$\sum_{i=1}^{3M} a_i[h_i(x)] = \cos \pi(x), \tag{85}$$

$$y(x) = \sum_{i=1}^{3M} a_i[q_i(x) - xq_i(1)] + x. \tag{86}$$

The HWCM solution of this example with $J = 4$ is given in Table 19. The results are compared with the exact solution and are found to be in good agreement. Figures 10. shows the comparison of the HWCM solution with the exact solution. The error estimates obtained for different J are given in Table 20.

5 Conclusion

In this paper, an efficient numerical scheme based on uniform 3-scale Haar wavelets is used to solve ordinary differential equations. The numerical scheme is tested for four examples. The obtained numerical results are compared with the exact solutions, and are found to be in good agreement. Also,the method does not require conversion of a boundary value problem into initial value problem by using shooting like procedure and hence has higher stability. Thus the Haar wavelet method guarantees the necessary accuracy with a small number of grid points.

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Table 1: Comparison of the HWCM solution and exact solution of Example 1

x	$y(x)$	
	HWCM	Exact
0.1	2.242506	2.242472
0.2	22.581038	2.580977
0.3	3.035316	3.035197
0.4	3.629782	3.629577
0.5	4.394701	4.394373
0.6	5.367430	5.366926
0.7	6.593979	60593232
08	8.130933	8.129852
0.9	10.047933	10.046261

Table 2: Error in the solution of Example 1

J	σ	
	L_2	L_∞
1	2.2117E-02	1.6450E-02
2	1.4042E-03	6.7504E-04
3	8.9982E-05	2.5982E-05
4	5.7717E-06	2.5982E-07
5	3.7024E-07	3.6280E-08

Table 3: Comparison of the HWCM solution and exact solution of Example 2

x	$y(x)$	
	HWCM	Exact
0.1	0.000004	0.000008
0.2	0.000153	0.000153
0.3	0.000835	0.000834
0.4	0.002834	0.002832
0.5	.007435	0.007430
0.6	0.016571	0.016562
0.7	0.033301	0.032999
08	0.060584	0.060561
0.9	0.104435	0.104405

Table 4: Error in the solution of Example 2

J	σ	
	L_2	L_∞
1	4.7630E-04	3.7412E-04
2	3.0534E-05	1.5791E-05
3	1.9586E-06	6.1241E-07
4	1.2564E-07	2.3038E-08
5	8.0602E-08	8.5774E-10

Table 5: Comparison of the HWCM solution and exact solution of Example 3

x	$y(x)$	
	HWCM	Exact
0.1	-0.115518	-0.115513
0.2	-0.264232	-0.264220
0.3	-0.449694	-0.449670
0.4	-0.675931	-0.675892
0.5	-0.947569	-0.947508
0.6	-1.269973	-1.269882
0.7	-1.649426	-1.649295
0.8	-2.093343	-2.093156
0.9	-2.610554	-2.610269

Table 6: Error in the solution of Example 3

J	σ	
	L_2	L_∞
1	3.7727E-03	2.7560E-03
2	2.3962E-04	1.1242E-04
3	1.5355E-05	4.3177E-06
4	9.8492E-07	1.6193E-08
5	6.3182E-08	6.0229E-09

Table 7: Comparison of the HWCM solution and exact solution of Example 4

x	$y(x)$	
	HWCM	Exact
0.1	1.120423	1.120703
0.2	1.073970	1.074340
0.3	0.867972	0.868229
0.4	0.534922	0.534912
0.5	0.127353	0.127011
0.6	-0.290443	-0.291075
0.7	-0.652569	-0.653339
0.8	-0.901903	-0.902588
0.9	-0.999116	-0.999471

Table 8: Error in the solution of Example 4

J	σ	
	L_2	L_∞
1	1.2430E-03	7.2401E-04
2	8.5697E-05	2.8579E-05
3	5.5402E-06	1.0661E-06
4	3.5570E-07	3.9532E-08
5	2.2820E-07	1.4643E-09

Table 9: Comparison of the HWCM solution and exact solution of Example 5

x	$y(x)$	
	HWCM	Exact
0.1	0.218577	0.218536
0.2	0.467278	0.467302
0.3	0.733998	0.734035
0.4	1.004249	1.004298
0.5	1.262147	1.262206
0.6	1.491194	1.491262
0.7	1.675192	1.799232
0.8	1.799915	1.799232
0.9	1.850326	1.850310

Table 10: Error in the solution of Example 5

J	σ	
	L_2	L_∞
1	1.4626E-03	6.5134E-04
2	9.4088E-05	2.4365E-05
3	6.0376E-06	9.0300E-07
4	3.8733E-07	3.3447E-08
5	2.4847E-08	1.2388E-09

Table 11: Comparison of the HWCM solution and exact solution of Example 6

x	$y(x)$	
	HWCM	Exact
0.1	0.069100	0.069098
0.2	0.147500	0.147497
0.3	0.234418	0.234414
0.4	0.328986	0.328981
0.5	0.430257	0.430253
0.6	0.537221	0.537217
0.7	0.648808	0.648805
0.8	0.763904	0.763902
0.9	0.881359	0881358

Table 12: Error in the solution of Example 6

J	σ	
	L_2	L_∞
1	5.9865E-03	2.7729E-03
2	3.4238E-03	9.1851E-04
3	1.9746E-03	3.0596E-04
4	1.1399E-03	1.0198E-04
5	6.5814E-04	3.3993E-05

Table 13: Comparison of the HWCM solution and exact solution of Example 7

x	$y(x)$	
	HWCM	Exact
0.1	0.018647	0.018641
0.2	0.036097	0.036097
0.3	0.051119	0.051119
0.4	0.062782	0.062782
0.5	0.069746	0.069746
0.6	0.071017	0.071018
0.7	0.065584	0.065585
0.8	0.052502	0.052502
0.9	0.030911	0.030901

Table 14: Error in the solution of Example 7

J	σ	
	L_2	L_∞
1	9.5161E-06	4.3872E-04
2	6.2344E-07	1.6735E-07
3	4.0086E-08	6.2131E-09
4	2.5721E-09	2.3017E-10
5	1.6510E-10	8.5253E-12

Table 15: Comparison of the HWCM solution and exact solution of Example 8

x	$y(x)$	
	HWCM	Exact
0.1	0.085232	0.085233
0.2	0.171318	0.171320
0.3	0.259119	0.259121
0.4	0.349514	0.349516
0.5	0.443406	0.443409
0.6	0.541737	0.541740
0.7	0.645489	0.645492
0.8	0.755703	0.755705
0.9	0.873480	0.873481

Table 16: Error in the solution of Example 8

J	σ	
	L_2	L_∞
1	5.9865E-03	2.7429E-03
2	3.4238E-03	9.1851E-04
3	1.9746E-03	3.0596E-04
4	1.1399E-03	1.0198E-04
5	6.5814E-04	3.3993E-05

Table 17: Comparison of the HWCM solution and exact solution of Example 9

x	$y(x)$	
	HWCM	Exact
0.1	0.155505	0.155512
0.2	0.313239	0.313252
0.3	0.475519	0.475538
0.4	0.644844	0.644869
0.5	0.823998	0.824027
0.6	1.016159	1.016189
0.7	1.225025	1.225055
0.8	1.454967	1.454992
0.9	1.711202	1.711217

Table 18: Error in the solution of Example 9

J	σ	
	L_2	L_∞
1	6.2923E-05	2.9588E-05
2	4.1457E-06	1.1193E-06
3	2.6673E-07	4.1618E-08
4	1.7116E-08	1.5420E-09
5	1.0980E-09	5.7113E-09

Table 19: Comparison of the HWCM solution and exact solution of Example 10

x	$y(x)$	
	HWCM	Exact
0.1	0.015296	0.015305
0.2	0.021166	0.021177
0.3	0.019015	0.019026
0.4	0.011039	0.011045
0.5	-0.000000	-0.000000
0.6	-0.011039	-0.011045
0.7	-0.019015	-0.019026
0.8	-0.021166	-0.021177
0.9	-0.015296	-0.015305

Table 20: Error in the solution of Example 10

J	σ	
	L_2	L_∞
1	2.3678E-05	1.0614E-05
2	1.6440E-06	4.3976E-07
3	1.0635E-07	1.6485E-08
4	6.8288E-09	6.1123E-10
5	4.3811E-10	2.2640E-11

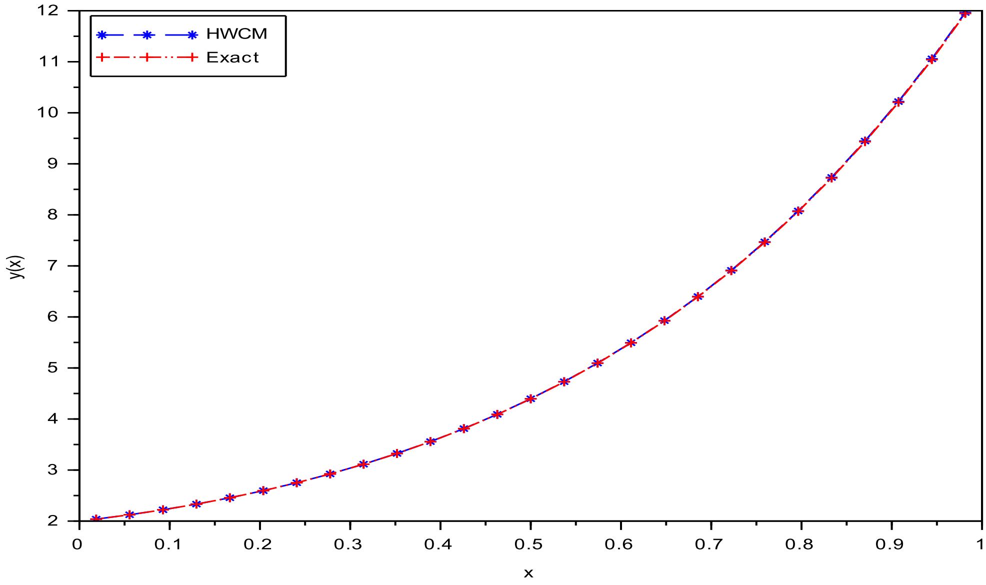

Figure 1: Comparison of HWCM solution and exact solution of Example 1.

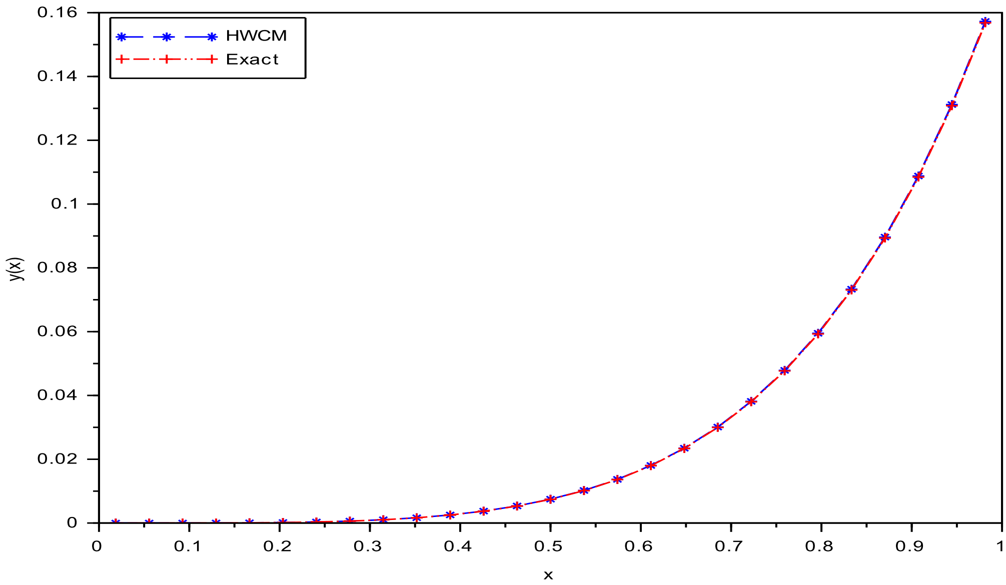

Figure 2: Comparison of HWCM solution and exact solution of Example 2.

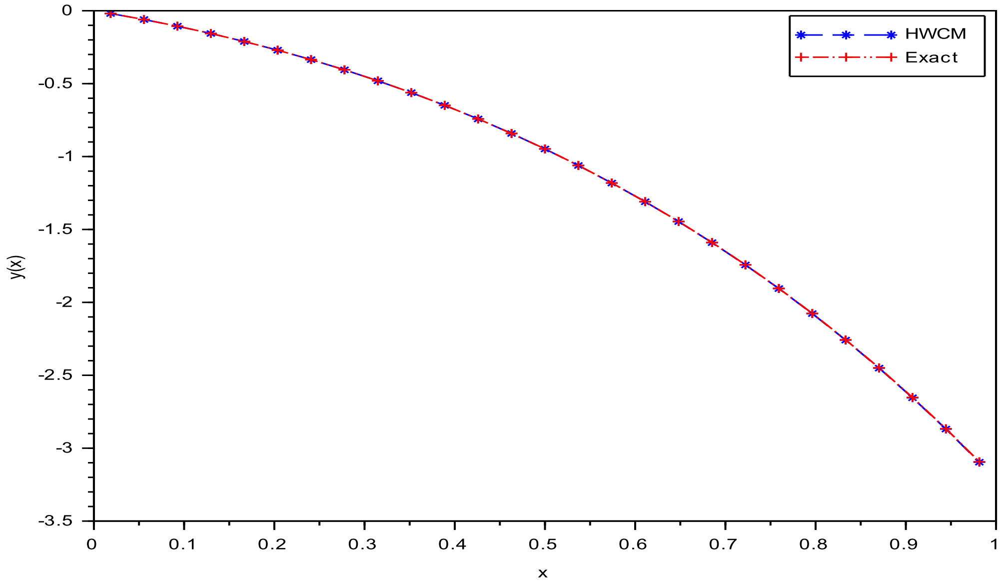

Figure 3: Comparison of HWCM solution and exact solution of Example 3.

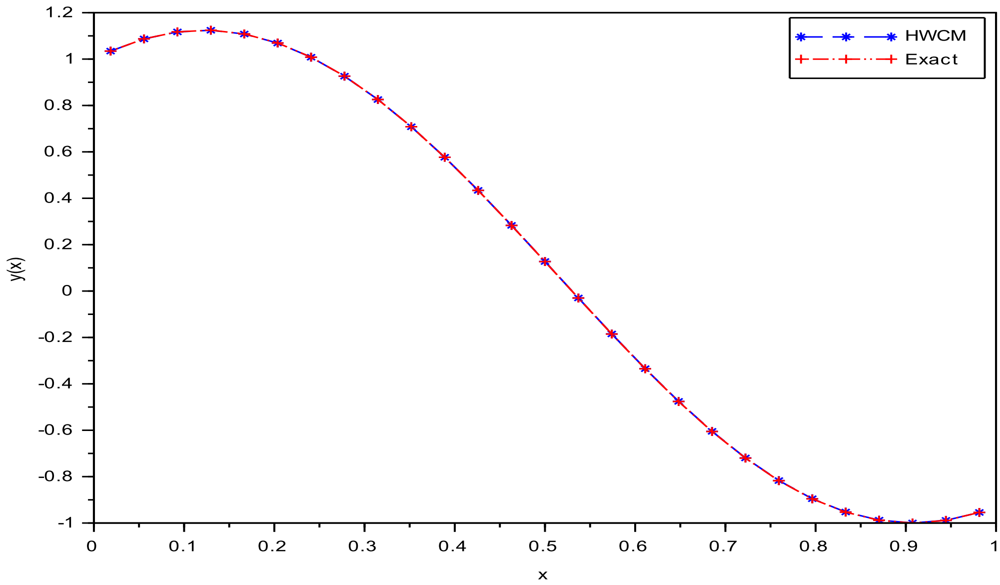

Figure 4: Comparison of HWCM solution and exact solution of Example 4.

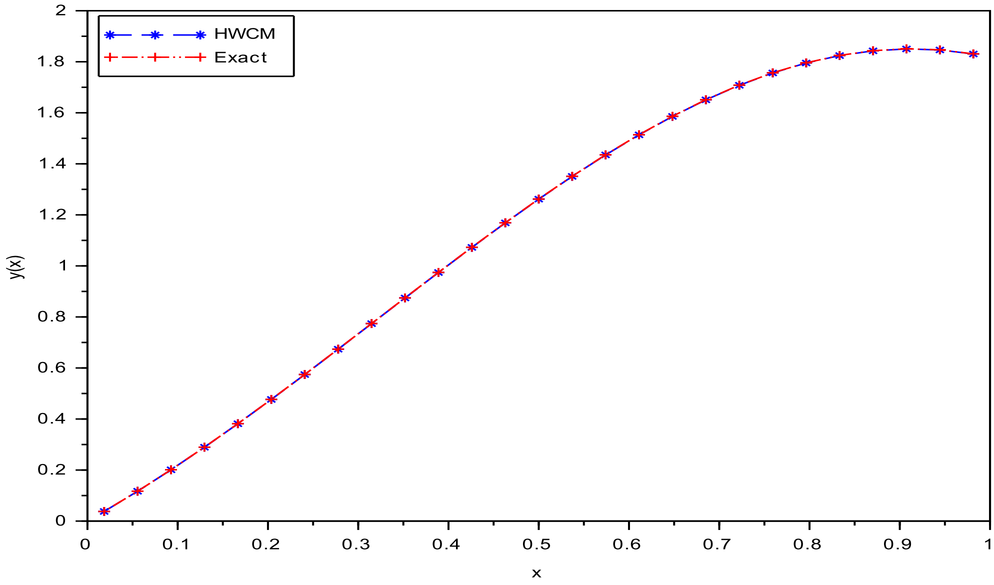

Figure 5: Comparison of HWCM solution and exact solution of Example 5.

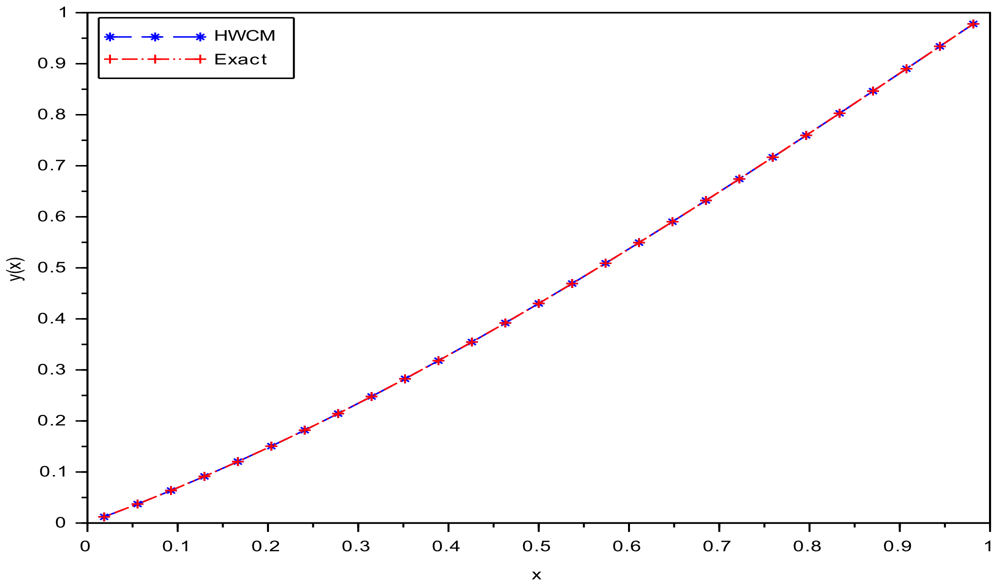

Figure 6: Comparison of HWCM solution and exact solution of Example 6.

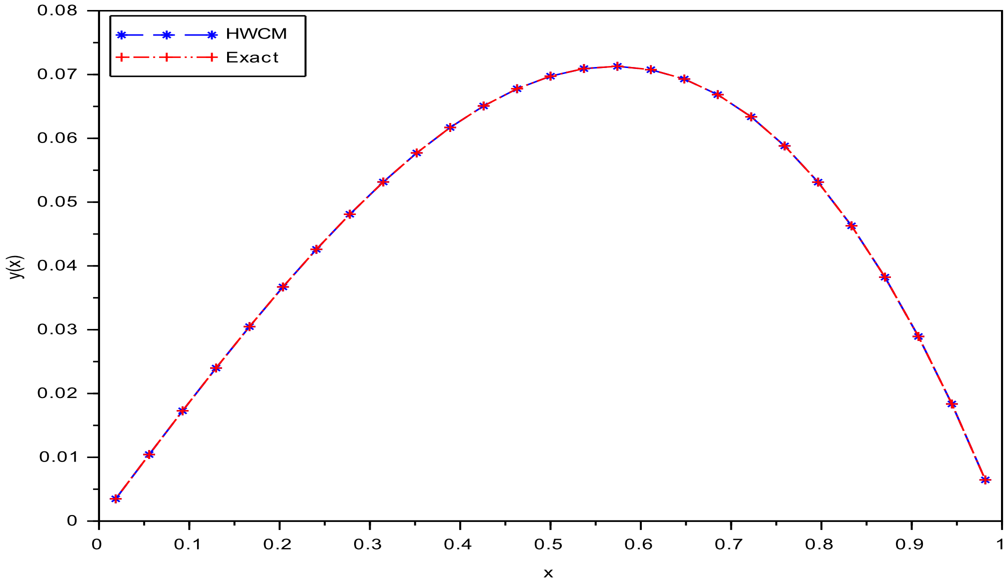

Figure 7: Comparison of HWCM solution and exact solution of Example 7.

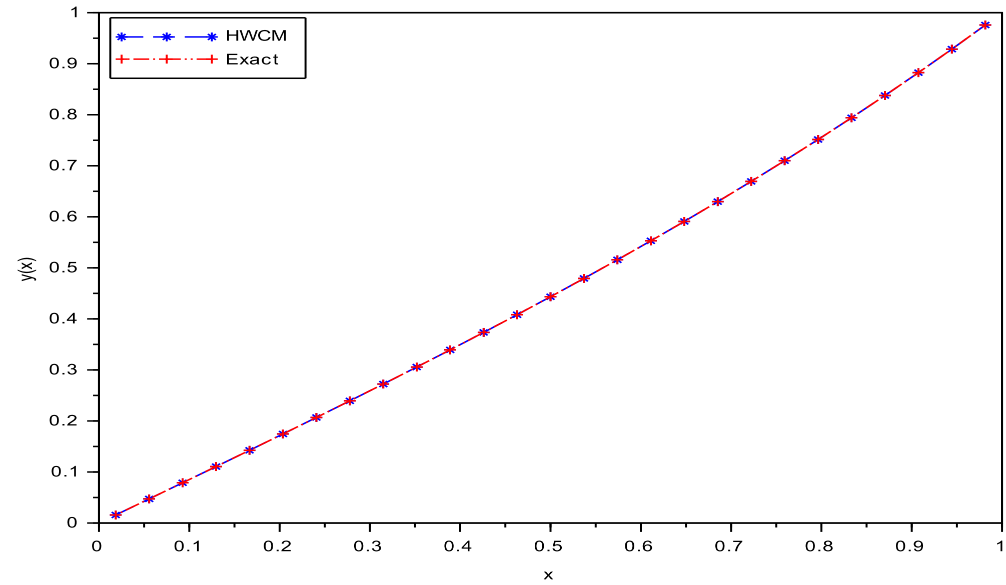

Figure 8: Comparison of HWCM solution and exact solution of Example 8.

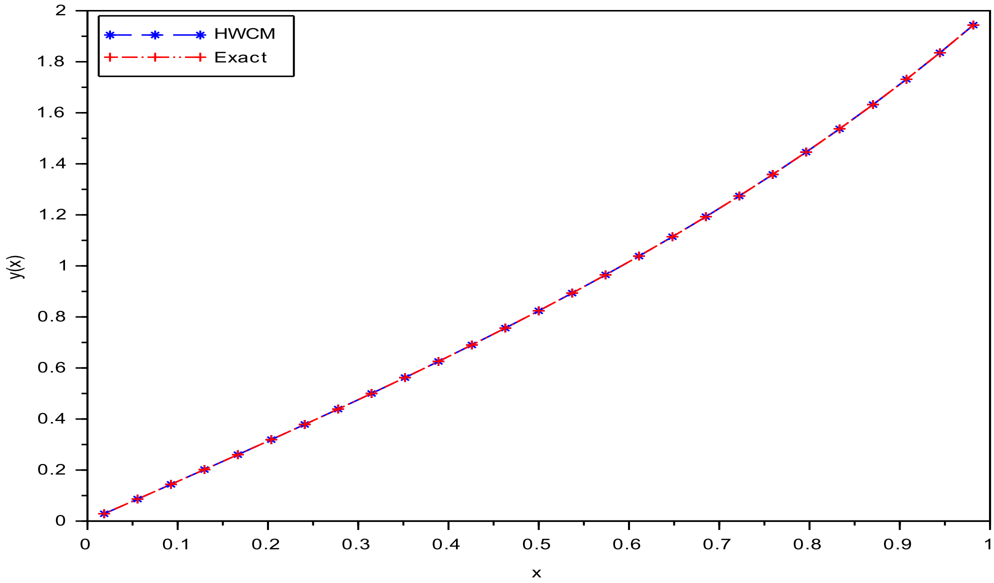

Figure 9: Comparison of HWCM solution and exact solution of Example 9.

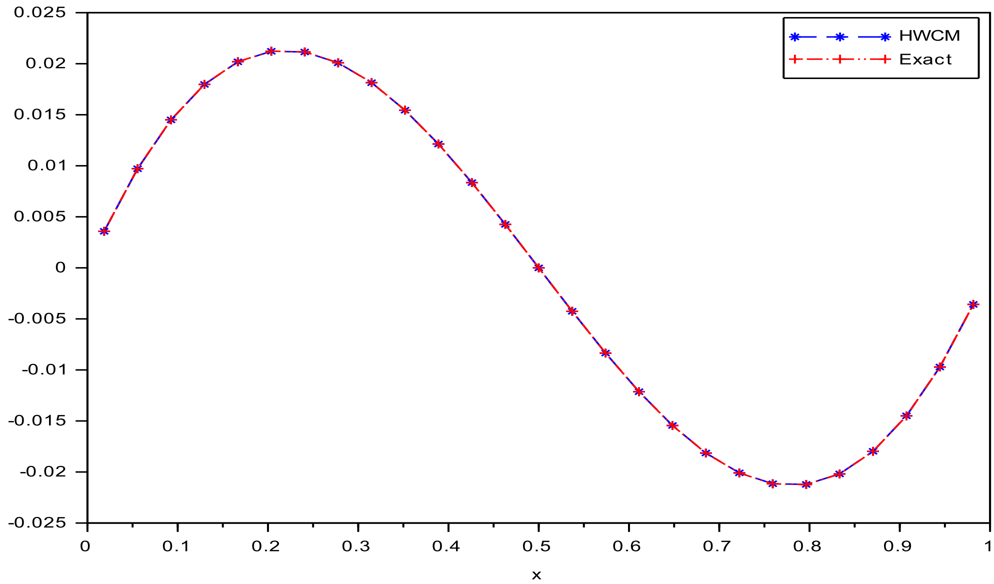

Figure 10: Comparison of HWCM solution and exact solution of Example 10.

Study of Quasi Linear Equations and Their Discontinuities

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Abstract: In this paper we study quasilinear equations of two and more than two variables and their discontinuities. We have studied Cauchy Kowalasky theorem and classical singularity propagation theorem for analysis of singularity and propagation of discontinuity of PDE.

Keywords: Cauchy problem, Quasilinear equation, Singularities, Discontinuities, Bicharacteristics, Propagation theorem.

1 Introduction

In mathematics, a singularity is generally a point at which a given mathematical object is not defined such a point is called as 'SINGULAR POINT' [5]. The behaviour of that object at that point can be explained by the solution of the governing differential equation obtained by modelling the situation. Thus the study of singularities is an important part of the study of a material object. Singular point is also called as not an ordinary point. Ordinary point is the point at which function is analytic.

1.1 Singularity of a function

The following are the examples for singularities of a function.

Example 1:

$$f(x) = \frac{1}{1-x}.$$

$f(x) \to \infty$ at $x = 1$,
$x = 1$ is singular point.

Example 2:

$$f(x) = \frac{1}{1-x^2}.$$

$f(x) \to \infty$ at $x = \pm 1$,
$x = \pm 1$ is singular points

Example 3:

$$f(x) = \frac{1}{x(1-x)}.$$

$f(x) \to \infty$ at $x = 0$ and $x = 2$
$x = 0$ and $x = 2$ are singular points.
$\therefore f(x)$ is analytic function along the real line expect at 0 and 2.

1.2 Singularity for Ordinary Differential Equation

Consider second order linear ordinary differential equation [9],

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x), \tag{1}$$

where $P(x)$ and $Q(x)$ functions are not defined at $x = x_0$ then that point is 'SINGULAR POINT'. This point is called as singularity of equation.

Example 4:

$$\frac{d^2y}{dx^2} + \frac{1}{1-x}\frac{dy}{dx} + \frac{1}{1-x}y = x,$$

where $P(x) = \frac{1}{1-x}$, $Q(x) = \frac{1}{1-x}$,
where $P(x)$ and $Q(x)$ are not analytic at $x = 1$ then, it is a 'Singular point'.

Example 5:

$$\frac{d^2y}{dx^2} + \frac{1}{x(1-x)}\frac{dy}{dx} + \frac{sin(x)}{x(1-x)}y = x,$$

where $P(x) = \frac{1}{x(1-x)}$, $Q(x) = \frac{sin(x)}{x(1-x)}$,
where $P(x)$ and $Q(x)$ are not analytic at $x = 0, 1$ then these are 'Singular points'.

1.3 Types of singularities

Singular points are broadly divided into two types namely [5],

1. **Regular singular points:** A point x_0 is said to be regular singular point if both $(x - x_0)P(x)$ and $(x - x_0)^2Q(x)$ are analytic at $x = x_0$
2. **Irregular singular points:** A point x_0 is said to be Irregular singular point if any one $(x - x_0)P(x)$, $(x - x_0)^2Q(x)$ are analytic at $x = x_0$.

1.4 Classification of Singularities

1.4.1 Isolated Singularity

A singular point will be an isolated point of the set, if the function is analytic at each point in some deleted neighbourhood of the point. That is a limiting point of the set, of singular points is itself a singular point so that the set of singular points is closed. Using the Laurent's series expansion, the various possibilities for the behaviour of a function near an isolated singular point can be explained [3, 5, 7, 9, 10, 11].

Suppose $z = a$ is an isolated singular point of a function $f(z)$ and $f(z)$ is analytic in the domain $0 < |z - a| < r$, where r is a positive number. Then by Laurent's series,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n,$$

where $a_n = \frac{1}{2\pi i}\int_e \frac{f(z)}{(z-a)^{n+1}}dz$, and C is a circle $|z - a| = \rho < r$,

here $a_{-1}(z-a)^{-1}+a_{-2}(z-a)^{-2}+\ldots+a_{-n}(z-a)^{-n}+\ldots$ of the Laurent's series consisting of the sum of negative powers of $(z-a)$ is called the principal part of $f(z)$ at $z=a$.
Relative to the principal part, there are three possibilities which consist of three cases

(a) no term.
(b) a finite number of terms with non-zero co-efficients.
(c) an infinite number of terms with non-zero co-efficients.

1.4.2 Removable Singularity

The principal part consists of no terms,

$$\text{for} \quad 0<|z-a|<r, \quad f(z)=a_0+a_1(z-a)+\ldots+a_n(z-a)^n+\ldots \tag{2}$$

$$\phi(z)=\begin{cases} f(z) & \text{for} \quad 0<|z-a|<r \\ a_0 & \text{for} \quad z=a \end{cases}$$

$\phi(z)$ being the sum function of a power series is analytic at $z=a$. Therefore Singularity of $f(z)$ at $z=a$ is removable. By defining the value of the function at a single point $z=a$, the singularity can disappear. Due to Riemann's theorem, we have "If $z=a$ is an isolated singularity of a function $f(z)$ and is bounded in some neighbourhood of a; then there exists one and only one complex number ξ such that the function which is defined as equal to $f(z)$ in some deleted neighbourhood of a and equal to ξ at a, is analytic at a" [9, 10, 11].

Let $|f(z)|<k \quad \text{for} \quad 0<|z-a|<r$ and C_ρ be any circle with its center at a and radius ρ. We know that, by Laurent's series

$$a_n=\frac{1}{2\pi i}\int_{C_\rho}\frac{f(z)}{(z-a)^{n+1}}dz.$$

Now we have

$$|a_n|\leq\frac{1}{2\pi}\frac{k}{\rho^{n+1}}2\pi\rho=\frac{k}{\rho^n}.$$

If n be a negative integer, then $a_n\to 0$ as $\rho\to 0$. Thus principal part of $f(z)$ consists of no non-zero coefficient so $f(z)$ has a removable singularity at $z=0$.
If $f(z)=a_0+a_1(z-a)+\ldots+a_n(z-a)^n+\ldots$, then the required value ξ is given by

$$\xi=a_0=\frac{1}{2\pi i}\int_{C_\rho}\frac{f(z)}{z-a}dz.$$

Example 6: If $f(z)=\dfrac{\sin(z)}{z}$, then $z=0$ is a removable singularity (since $f(0)$ is not defined),

$$\text{but} \lim_{z\to 0}\frac{\sin(z)}{z}=1,$$

$$\text{we define } f(0)=\lim_{z\to 0}\frac{\sin(z)}{z}=1,$$

$$\therefore \frac{\sin(z)}{z}=\frac{1}{z}\left\{z-\frac{z^3}{3!}+\frac{z^5}{5!}\mp\ldots\right\}=1-\frac{z^2}{3!}+\frac{z^4}{5!}\mp\ldots.$$

1.4.3 Poles

When the principal part consists of only a finite number of non-zero co-efficients (say n) of negative powers of $(z-a)$. Let the principal part of $f(z)$ for $z=a$ be given by

$$\begin{aligned}\phi(z) &= a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \ldots + a_{-m}(z-a)^{-m}, \\ &= (z-a)^{-m}[a_{-m} + a_{-m+1}(z-a) + \ldots + a_1(z-a)^{m-1}],\end{aligned}$$

where $a_{-m} \neq 0$. Also $\psi(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots$. Thus

$$\begin{aligned}f(z) &= \psi(z) + (z-a)^{-m}[a_{-m} + a_{-m+1}(z-a) + \ldots + a_{-1}(z-a)^{m-1}] \\ &= (z-a)^{-m}[a_{-m} + a_{-m+1}(z-a) + \ldots + a_{-1}(z-a)^{m-1} + \psi(z)(z-a)^m] \\ &= (z-a)^{-m}F(z) \text{ (say).}\end{aligned}$$

where $F(z)$ is analytic in $|z-a| < r$ and $F(a) = a_{-m} \neq 0$. Thus the principal part of $f(z)$ consists of a finite number of non-zero terms only the last non-zero co-efficient is that of $(z-a)^{-m}$ then $f(z)$ can be expressed as

$$f(z) = (z-a)^{-m}F(z), \quad 0 < |z-a| < r \tag{3}$$

where $F(z)$ is analytic at $z=a$ and $F(a) \neq 0$. Therefore $z=a$ is a pole of order m of the function $f(z)$. From (3) we can write $\frac{1}{f(z)} = (z-a)^m \frac{1}{F(z)} = (z-a)^m G(z)$ where $G(z)$ is analytic at $z=a$ and $G(z) \neq 0$. Clearly $\frac{1}{f(z)}$ has a removable singularity at $z=a$.
If $\frac{1}{f(z)} \neq 0$ at $z=a$ then $\frac{1}{f(z)}$ has a zero of order m and hence $f(z)$ has a pole of order m for $z=a$.
Thus a pole of a function is a zero of the same order of the reciprocal of the function. Also zero of any order of a function is a pole of the same order of the reciprocal of the function [4].

1.4.4 Essential Singularity

The principal part consists of an infinite number of terms with non-zero co-efficients. Also an isolated singular point is called an essential singularity if the singularity is neither removable nor a pole [3, 5, 7, 9]. Let us quote and prove Weierstrass theorem.

Weierstrass Theorem: "If $z=a$ is an essential singularity of a function, then for any arbitrary number η, arbitrary $\epsilon > 0$ and arbitrary $\rho > 0$, there exists a point z such that $0 < |z-a| < \rho$ for which $|f(z) - \eta| < \epsilon$." This means that in every arbitrary neighbourhood of an essential singularity, there exists a point for which the value of the function is arbitrarily near any arbitrarily assigned number.

Proof: We prove the theorem by the method of contradiction.
Suppose the result is not true, i.e., $\exists$ a number η and a deleted neighbourhood $0 < |z-a| < r$ for every point z of which $|f(z) - \eta| > \epsilon$
where ϵ is a given positive number.
Thus for $0 < |z-a| < r$, we have $|\frac{1}{f(z)-\eta}| < \frac{1}{\epsilon}$.
Applying Riemann's theorem,
$\frac{1}{f(z)-\eta}$ has a removable singularity at $z=a$.
Suppose in the neighbourhood of $z=a$, we have

$$\frac{1}{f(z)-\eta} = C_0 + C_1(z-a) + C_2(z-a)^2 + \ldots$$

Now if $C_0 \neq 0$ and if we define $f(z)$ for $z = a$,
by the equality $\frac{1}{f(z)-\eta} = C_0$ i.e., $f(a) = \eta + \frac{1}{C_0}$.
$\frac{1}{f(z)-\eta}$ becomes analytic and non-zero for $z = a$.
$f(z)$ is itself analytic at $z = a$.
This is a contradiction.
Again suppose $C_0 = C_1 = \ldots = C_{m-1} = 0,\ C_m \neq 0$
$z = a$ is a zero of order m of $\frac{1}{f(z)-\eta}$,
$z = a$ is a pole of order m of $[f(z) - \eta]$,
$z = a$ is a pole of $f(z)$.
Thus again this is a contradiction.
Hence the Weierstrass theorem stated above is true.

Example 7: Since $e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \ldots$
$z = 0$ is an essential singularity.

1.4.5 Singularities at infinity

Suppose $w = f(z)$ is defined in some neighbourhood of infinity.
Let the transformation $z = \frac{1}{z}$.
Consider the function $w = f(\frac{1}{z}) = \phi(z)$(say) in a neighbourhood of $z = 0$, surely $\phi(z)$ is not defined for $z = 0$.
$\therefore$ $f(z)$ is analytic at ∞ if $\phi(z)$ has a removable singularity at $z = 0$. Further $f(z)$ is said to have a pole of order m or an essential singularity at ∞.
Similarly $\phi(z)$ has a pole of order m or an essential singularity at $z = 0$.

Example 8: $f(z) = z^3$ has a pole of order 3 at $z = \infty$,
since $F(w) = f(\frac{1}{w}) = \dfrac{1}{w^3}$ has a pole of order 3 at $w = 0$.
Similarly $f(z) = e^z$ has an essential singularity at $z = \infty$,
since $F(w) = f(\frac{1}{w}) = e^{\frac{1}{w}}$ has an essential singularity at $w = 0$.

1.4.6 Residue

Let $z = a$ be an isolated singularity of $f(z)$.
Let $f(z)$ be a single valued and analytic inside and on a circle C except at the point $z = a$, which is the center of C. i.e., $f(z)$ is analytic in a domain $0 < |z - a| < r$ for some $r > 0$.
Thus $\dfrac{1}{2\pi i}\displaystyle\int_C f(z)dz$ is called the Residue of $f(z)$ at $z = a$, where C is a positively oriented simple closed curve with center at a and radius $\rho < r$ [9, 7].
Also the residue of $f(z)$ or a finite isolated singular point $z = a$ is the co-efficient of $(z - a)^{-1}$ in the Laurent's expansion of $f(z)$ around $z = a$. This is given by

$$\begin{aligned} f(z) &= \sum_{n=-\infty}^{\infty} a_n(z-a)^n, \qquad (4) \\ &= a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \ldots \end{aligned}$$

where

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}dz, \quad n = 0, \pm 1, \pm 2, \ldots \qquad (5)$$

In special case $n = -1$, we have

$$a_{-1} = \frac{1}{2\pi i}\oint_C f(z)dz,$$

$$\oint_C f(z)dz = 2\pi i a_{-1}. \tag{6}$$

$$\oint_C \frac{dz}{(z-a)^p} = \begin{cases} 2\pi i & p = 1 \quad (\text{inside}C) \\ 0 & p = \text{integer} \neq 1 \quad (\text{outside}C) \end{cases} \tag{7}$$

Since (6) involves only the co-efficients a_{-1}, a_{-1} is called the residue of $f(z)$ at $z = a$.

2 Introduction to Cauchy Problem

To find the solution of the partial differential equations there are many methods. Method of characteristics is one of the method.[1, 3, 7]
The general first order partial differential equation of n variables $(x_1, x_2, x_3, ..., x_n)$ of the form

$$f(x_1, x_2, x_3, \cdots, x_n, u, u_{x_1}, u_{x_2}, \cdots, u_{x_n}) = 0, \tag{8}$$

the general solution of an equation (8) can be obtained by solving the Ordinary Differential equations. This is not true for higher order equations.
Consider first order of partial differential of two variables x and y

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u), \tag{9}$$

where functions a and b depends on x and y only, then equation (9) is called **Semilinear equation**

$$a(x, y)u_x + b(x, y)u_y = c(x, y, u). \tag{10}$$

The functions a and b depends on x, y and u, then (9) is called **Quasilinear equation**

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u). \tag{11}$$

The solution $u = u(x, y)$ represent a surface in (x, y, u) space. This surface is called an integral surface of partial differential equation. Almost always we deal with those solution of partial differential equation which satisfy certain conditions, such conditions can be formulated as a Cauchy problem.

3 The Cauchy Problem for Second Order Partial Differential Equations

A Cauchy problem for solution of a partial differential equation that satisfies certain condition that are given on a surface in the domain D [1, 3, 7].
Consider a general **quasi-linear** second order equation with two independent variable x and y

$$au_{xx} - 2bu_{xy} + cu_{yy} = d, \tag{12}$$

where a, b, c and d function of x and y and u the Cauchy problem consist a solution of equation (12) with given value of u and its normal derivative on a curve C in the (x, y)-plane.
Let the parametric representation of C be

$$\begin{array}{l} x = x_0(s) \\ y = y_0(s), s \in I \end{array}, \tag{13}$$

where I is in an interval on the real line.
The solution of Cauchy problem which satisfying the some conditions

$$\begin{array}{l} u(x, y) = u_0, \\ u\left(x_0(s), y_0(s)\right) = u_0(s), \\ \dfrac{\partial u}{\partial \vartheta}\left(x_0(s), y_0(s)\right) = u_1(s), \end{array} \tag{14}$$

where $\dfrac{\partial}{\partial \vartheta}$ denote a normal derivative to C and we assume a, b, c and d are analytic function regular in given domain D. First a solution is to show that the partial derivative of u of all order are uniquely determined at every point of curve C, differentiate partially equation (14) with respect to s,

$$\frac{\partial u}{\partial x_0}\frac{\partial x_0}{\partial s} + \frac{\partial u}{\partial y_0}\frac{\partial y_0}{\partial s} = \frac{\partial u_0(s)}{\partial s}. \tag{15}$$

Equation (15) can also be written in the form of

$$x_0' u_{x_0} + y_0' u_{y_0} = u_0'(s),$$

where a prime ($\prime$) denotes differentiation with with respect to s. Except at point where x_0 and y_0 vanish u_{x_0} and u_{y_0} can be determine uniquely.
Now second order derivative, namely $u_{xx_0}(s)$ and $u_{xy_0}(s)$ and $u_{yy_0}(s)$,

$$\begin{array}{r} a u_{xx_0}(s) + 2b u_{xy_0}(s) + c u_{yy_0}(s) = d \\ x_0'(s) u_{xx_0}(s) + y_0'(s) u_{xy_0}(s) = \{u_{x_0}(s)\}' \\ x_0'(s) u_{xy_0}(s) + y_0'(s) u_{yy_0}(s) = \{u_{y_0}(s)\}'. \end{array} \tag{16}$$

Now equation (16) can be written in form of matrix is then, we get

$$\begin{bmatrix} a & 2b & c \\ \dfrac{dx_0}{ds} & \dfrac{dy_0}{ds} & 0 \\ 0 & \dfrac{dx_0}{ds} & \dfrac{dy_0}{ds} \end{bmatrix} = \begin{bmatrix} d \\ \{u_{x_0}(s)\}' \\ \{u_{y_0}(s)\}' \end{bmatrix}.$$

The determinant of the coefficient matrix is non zero

$$\begin{vmatrix} a & 2b & c \\ \dfrac{dx_0}{ds} & \dfrac{dy_0}{ds} & 0 \\ 0 & \dfrac{dx_0}{ds} & \dfrac{dy_0}{ds} \end{vmatrix} \neq 0,$$

$$a(y_0')^2 - 2b x_0' y_0' + c(x_0')^2 \neq 0. \tag{17}$$

We can show that the derivative of u of all order can be uniquely determined at point of C, provided

$$a(y_0')^2 - 2bx_0'y_0' + c(x_0')^2 \neq 0.$$

In this way we can formally develop a unique Taylor's series expansion solution in the neighborhood of point of C, satisfying the given conditions on C. First show that Taylor's expansion is convergent to curve C, the difficulty is to show that such an expansion is convergent in some region around C.

The Cauchy-Kowalewski Theorem: There exist a solution of the Cauchy problem which is analytic at x^0 given by a power series in $x - x^0$ and that there is no other analytic solution [1, 7].
The theorem consists of all coefficients for a prospective power series solution u at x^0 can obtained by successive differentiation form the differential equation and Cauchy data, and that the series convergent to a solution.
If

$$a(y_0')^2 - 2bx_0'y_0' + c(x_0')^2 = 0,$$

the partial derivative of u on the curve c can not be determined uniquely are prescribed, no unique solution of equation (12) The curve c: $x = x_0(s)$, $y = y_0(s)$ in (x, y)-plane is given by the equation

$$\xi(x, y) = \text{constant}, \tag{18}$$

$$d\xi = \frac{\partial \xi}{\partial x}dx + \frac{\partial \xi}{\partial y}dy,$$
$$\frac{\xi_x}{\xi_y} = -\frac{dy}{dx}$$

4 The Classical Singularity Propagation Theorem

Consider a linear partial differential equation of first order in the space (x, y) [6],

$$a(x, y)\frac{\partial u}{\partial x} + b(x, y)\frac{\partial u}{\partial x} = c(x, y), \tag{19}$$

where a, b, c are C^∞ function and $a^2 + b^2 \neq 0$.
The characteristic equation of equation (19), we have

$$\frac{dx}{ds} = a(x, y), \quad \frac{dy}{ds} = b(x, y), \quad \frac{du}{ds} = c(x, y). \tag{20}$$

Suppose that there is a curve

$$\ell : x = \xi(t), \quad y = \eta(t), \quad u = \zeta(t), \tag{21}$$

where $\xi(t)$, $\eta(t)$, $\zeta(t)$ are continuously differential and $\xi_t^2 + \eta_t^2 \neq 0$. The $u(x, y)$ is the solution of equation (19) and $u(\xi(t), \eta(t)) = \zeta(t)$ is called Cauchy problem.
To solve the characteristic equation (20) with initial data

$$x(0) = \xi(t), \; y(0) = \eta(t). \tag{22}$$

The solution of the above initial value problem are a family of curves with one parameter t:

$$x = \xi(s, t), \; y = \eta(s, t). \tag{23}$$

From equation (20) and equation (21)

$$\frac{du}{ds} = c(x(s,t), y(s,t)), u|_{s=0} = \zeta(t), \tag{24}$$

then $u(s,t)$ is the solution of equation (24).
There by using the inverse transformation of equation (23), we obtain the solution $u(x, y)$ of problem equation (19). Since the functions a, b only depends on $x,\ y$, then the procedure of solving the above two initial value problems is equivalent to solve the following problem

$$\left.\begin{array}{l} \dfrac{dx}{ds} = a, \quad \dfrac{dy}{ds} = b, \quad \dfrac{du}{ds} = c, \\ x|_{s=0} = \xi(t),\ y|_{t=0} = \eta(t),\ u|_{s=0} = \zeta(t). \end{array}\right\} \tag{25}$$

We have introduced characteristics as possible branch curves of integral surface along which certain derivatives of u have discontinuities. The magnitude of such jump is controlled by an ordinary differential equation of first order along the characteristic curves.

4.1 Analysis of Singularity

The characteristic curves are closely associated with the propagation of certain types of singularities. Along a non characteristics curves the Cauchy data uniquely determine the second derivative of a solution one approach to defining generalized solution equation (19) not necessarily of class C^2, consist of considering that have jump discontinuities along a curve [1, 2, 6, 8]. Singularity is the antonym of regularity. In category of C^∞, any point, where the solution is not C^∞, can be considered as singular point. Therefore, even for a continuously differentiable function $u(x, y)$ one can also discuss the set of its singular points. Suppose that the solution $u(x, y)$ of equation (19) is C^∞ at point (x_1, y_1), then one can draw a C^∞ curve $\ell : x = \xi_1(t), y = \eta_1(t)$ Through (x_1, y_1), such that the tangential direction of ℓ is transversal to the vector field everywhere. Denote $\zeta_1(t) = u(\xi_1(t), \eta_1(t))$ is nothing but $u = u(x, y)$.t Then the functions $x(s,t), y(s,t)$ obtained by using above mentioned procedure are C^∞. The Jacobian,

$$\frac{\partial(x, y)}{\partial(x, y)} = a\eta_1'(t) - b\xi_1'(t) \neq 0.$$

This implies that $u(x, y)$ is C^∞ at any point along all characteristics through $(x_1, y_1, u(x_1, y_1))$. Conversely, if $u(x, y)$ is not C^∞ at (x_2, y_2), then $u(x, y)$ is also not C^∞ along the whole characteristics through (x_2, y_2) due to the uniqueness of the initial value problem of ordinary differential equation. In other word, the C^∞ singularity of differentiable solution propagates along characteristics of the solution. For piecewise smooth solution of equation (19) has weak discontinuity on the curve, then the curve must be characteristic of the equation.

Example 9: Consider the Cauchy problem of wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \tag{26}$$

satisfying the initial conditions

$$u(0, x) = 0, u_t(0, x) = \phi(x), \tag{27}$$

$$\phi(x) = \begin{cases} (x^2-1)^2, \ |x| \leq 1; \\ 0, \ |x| \geq 1. \end{cases} \tag{28}$$

Then solution of Cauchy problem equation (27),(28) is

$$u(x,t) = \begin{cases} \frac{1}{2}[(x-at)^2-1]^2, \quad \text{if} \quad |x-at| \leq 1, \ |x+at| > 1; \\ \frac{1}{2}[(x+at)^2-1]^2, \quad \text{if} \quad |x-at| > 1, \ |x+at| \leq 1; \\ \frac{1}{2}[(x-at)^2-1]^2 + \frac{1}{2}[(x+at)^2-1]^2, \quad \text{if} \quad |x-at \leq 1, \ |x-at \leq 1; \\ 0, \text{ Otherwise.} \end{cases} \tag{29}$$

The singularity of the solution $u(t,x)$ occurs on the characteristics through $(0,\pm 1)$. That is, singularity propagates along characteristics.

Consider a linear partial differential equation of order m,

$$\sum_{\alpha_1+\alpha_2......+\alpha_n=\alpha, \ |\alpha|\leq m} a_{\alpha_1,...,\alpha_n}(x) \frac{\partial^\alpha}{\partial {x_1}^{\alpha_1}....\partial {x_n}^{\alpha_n}} = f(x_1,...,x_n). \tag{30}$$

4.2 Weakly discontinuous solution

Assume that there is a C^∞ surface S in the domain Ω , where the function $u(x_1,....,x_n)$ is defined. If $u \in C^\infty(\Omega/S)$ is a solution of equation (30), $u \in C^{m-1}(\Omega)$ and the m^{th} derivative of u has discontinuity of first class on S, then u is called **weakly discontinuous solution** of equation (30).
$\phi(x_1,....,x_n) = 0$ in equation (30) is called its characteristic surface, if the equality

$$\sum_{\alpha_1+\alpha_2......+\alpha_n=\alpha, \ |\alpha|\leq m} a_{\alpha_1,...,\alpha_n}(x) (\frac{\partial \phi}{\partial x_1})^{\alpha_1} .....(\frac{\partial \phi}{\partial x_n})^{\alpha_n} = 0, \tag{31}$$

is satisfied at each point on $\phi = 0$.

Theorem 1: If $u(x_1,....,x_n)$ is a weakly discontinuous solution of equation (30) , then he surface bearing the weak discontinuity of u must be a characteristic surface.
Note: The above theorem assert that the discontinuity of m^{th} derivatives of solution to equation (30) is always distributed on the characteristic surface. The facts is also valid for weaker singularities. That is, if $u(x-1,...,x_n)$ is a C^k solution of equation (30) with $k \geq m$, and the $(k+1)^{th}$ derivatives has discontinuity of first class on S, Then S must be characteristics of equation (30). The above theorem hold for also non linear equation.
Consider a non linear partial differential equation of higher order

$$F(x_1,...,x_n,u,..., \frac{\partial^{\alpha_1+....+\alpha_n} u}{\partial {x_1}^{\alpha_1}......\partial {x_n}^{\alpha_n}}) = 0, \tag{32}$$

where F is a C^∞ function of its variables $x_1,....,x_n,u,..,p_{\alpha_1.....\alpha_n}$. For a given solution $u(x)$, a surface $\psi(x_1,...,x_n) = 0$ is called characteristic surface, if the the following equality is satisfied,

$$\sum_{|\alpha_1+\alpha_2......+\alpha_n|=m} \frac{\partial F}{\partial p_{\alpha_1,....,\alpha_n}} (x,u,...., \frac{\partial^{\alpha_1+....+\alpha_n}}{\partial {x_1}^{\alpha_1},....,\partial {x_n}^{\alpha_n}}) (\frac{\partial \psi}{\partial x_1})^{\alpha_1} .....(\frac{\partial \psi}{\partial x_n})^{\alpha_n} = 0. \tag{33}$$

Theorem 2: If $u(x_1, ..x_n)$ is the C^∞ solution of equation (32) with $k \geq m+1$, and its derivative of $(k+1)^{th}$ order have discontinuity of first class on S, then S must be the characteristic surface of equation (32).

Note: In quasilinear or fully non linear case Since the characteristic surface S depends on the solution u, then the precise information on the singularity of solutions can only be obtained when the solution is obtained. This is different from the case of linear or semi linear equations. In the case the characteristics are independent of solutions.

4.2.1 Bicharacteristics

About equation (31) as a partial differential equation of first order, its characteristics is called **Bicharacteristics** of the equation (30).

Denote equation (31) as $H(x_1, ..., x_n, \dfrac{\partial \phi}{\partial x_1}, ..., \dfrac{\partial \phi}{\partial x_n}) = 0$, it is a non linear partial differential equation of the function ϕ. in the left hand side the function ϕ itself does not explicitly appear. For a given solution ϕ, the solution of the system of ordinary differential equations

$$\frac{dx_i}{ds} = \frac{\partial H}{\partial p_i}, \tag{34}$$

satisfying the initial condition $x_i(0) = x_{i0}$ is called the characteristics of equation (31). Denote $p = \dfrac{\partial \phi}{\partial x_i}(x_1, ..., x_n)$ on the surface $\phi = 0$, we have

$$\frac{dp_i}{ds} = \sum_j \frac{\partial^2 \phi}{\partial x_i \partial x_j} \frac{dx_j}{ds} = \sum_j \frac{\partial^2 \phi}{\partial x_i \partial x_j} H_{p_i}.$$

differentiating equation (31), we get

$$H_{x_i} + \sum_j H_{p_j} \frac{\partial^2 \phi}{\partial x_i \partial x_j} = 0. \tag{35}$$

Hence $p_i(s)$ satisfies

$$\frac{dp_i}{ds} = -\frac{\partial H}{\partial p_i}(x, \phi_x). \tag{36}$$

Therefore, $(x_1(s), ..., x_n(s), p_1(s), ..., p_2(s))$ satisfies the system

$$\left.\begin{aligned} \frac{dx_i}{ds} &= \frac{\partial H}{\partial p_i} \\ \frac{dp_i}{ds} &= -\frac{\partial H}{\partial x_i} \qquad (i = 1, ..., n), \end{aligned}\right\} \tag{37}$$

where the variables of H is $x_1, ..., x_n$ and $p_1, ..., p_n$. The solution of equation (37) is also called **Bicharacteristic strip** of equation (30).

Theorem 3: The singularity of weakly discontinuous solution to equation (30) on the characteristic surface $\phi = 0$ propagates along bi-characteristics.

Example 10: Consider three dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial {x_1}^2} - \frac{\partial^2 u}{\partial {x_2}^2} - \frac{\partial^2 u}{\partial {x_3}^2} = 0. \tag{38}$$

If $\phi(t, x_1, x_2, x_3) = 0$ is its characteristic surface, then the function ϕ should satisfy

$$\phi_t^2 - \phi_{x_1}^2 - \phi_{x_2}^2 - \phi_{x_3}^2 = 0. \tag{39}$$

When $\phi_t \neq 0$, one can solve $\phi(t, x_1, x_2, x_3) = 0$ to obtain $t = \psi(x_1, x_2, x_3)$, where ψ satisfies

$$\psi_{x_1}^2 + \psi_{x_2}^2 + \psi_{x_3}^2 = 1. \tag{40}$$

If $\phi(t, x_1, x_2, x_3) = 0$ is a surface bearing the singularities of u, then for any fixed t, the equation gives the location of the surface in (x_1, x_2, x_3) space at time t. The characteristics of equation (40) can be obtained from the projection of integral curves

$$\frac{dx_i}{ds} = 2p_i, \quad \frac{dp_i}{ds} = 0 \tag{41}$$

on the space (x_1, x_2, x_3). Arbitrarily taking a point $(x_{10}, x_{20}, x_{30}, p_{10}, p_{20}, p_{30})$, satisfying $\sum p_{i0}^2 = 1$, then the solution of equation (41) with the initial data

$$x_i(0) = x_{i0}, \quad p_i(0) = p_{i0} \quad (i = 1, 2, 3) \tag{42}$$

is

$$x_i = p_{i0}s + x_{i0}, \quad p_i = p_{i0}. \tag{43}$$

It is straight line with direction (p_{10}, p_{20}, p_{30}). Since $(\sum(x_i - x_{i0})^2)^{\frac{1}{2}} = s$, and (p_{10}, p_{20}, p_{30}) is the normal direction of the surface equation (40), then Theorem 3 indicates that the weak singularity of the solution equation (38) propagates along the normal direction of the wave front $\phi(x_1, x_2, x_3) = t$, and the speed of propagation is constant. Therefore, if the wave equation is applied to describe optical waves, then the equation $\phi(x_1, x_2, x_3) = t$ gives the motion of the front of optical waves, and equation (43) indicates that the ray of light propagates along straight line in homogeneous media.

Consider the motion of optical waves in inhomogeneous media,

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{n^2}\left(\frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {x_2}^2} + \frac{\partial^2 u}{\partial {x_3}^2}\right) = 0, \tag{44}$$

where $n = n(x_1, x_2, x_3)$ is the index of refraction. Then function ϕ satisfies

$$\phi_{x_1}^2 + \phi_{x_2}^2 + \phi_{x_3}^2 = n(x_1, x_2, x_3)^2. \tag{45}$$

In accordance, its characteristics satisfies

$$\frac{dx_i}{ds} = 2p_i, \quad \frac{dp_i}{ds} = (n^2)_{x_i}, \tag{46}$$

where p_i is not constant because $n(x_1, x_2, x_3)$ is not constant. Therefore, the characteristics are not straight lines any more. This means that the light does not propagate along straight line in inhomogeneous media.

5 Conclusion

The detailed study of types of singularities with classification is given in introduction. We have also proved Weierstrass theorem quoting Riemann theorem. Detail explanation about propagation of singularities is carried out using two important theorems. Bicharacteristics are also studied.

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Guided Teaching: An Innovative Design to Induce Critical Thinking in Students

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Abstract: Over the last two centuries, there has been tremendous progress in scientific knowledge and it is a challenging job of a teacher to prepare the younger generation to be able to imbibe that huge amount of knowledge and contribute to the future development of science. But the present science teaching methods rely more on loading the accumulated information to the students and little effort is made towards inducing research-provoking thoughts. This is one of the reasons why most of the students do rote learning to pass the examination. This study is focused on assessing effectiveness of an innovative method of science teaching in which the teacher acts as a guide and helps the students to gather information themselves and raise questions independently. It is strongly felt that innovative methods of teaching science may help in preparing students for future challenges in the field of science.

Keywords: Critical, thinking, learning, research, aptitude, guidance.

1 Introduction

In science teaching-learning process, critical thinking is of prime importance. It is the tendency of a student to think critically on a problem that helps it to find a solution. It also imparts the necessary confidence in a learner to search for solutions of problems of higher complexity. It is because of this reason that in all the organizations now-a-days candidates who can think critically are the most sought after candidates. Therefore, it becomes very important that we should use innovative methods during teaching so that the students get opportunity to think critically on a given problem.

It is a well-known fact that experiential learning is much more long-lasting than rote learning. Therefore, a lot of emphasis is put on learning-by-doing method now-a-days. Previous research papers have proven that there can be many low-cost materials available which can be utilized in learning-by-doing method. Now, the next step is to organize the materials in such a way that it should induce research-provoking thoughts. It is often noted that though learning-by-doing method is interesting for students, it often ends up in fun activity. The effectiveness of science teaching lies in the fact that it not only helps the learner to understand already established facts, but also induces a new chain of thoughts which leads to improvisation of previously established principles. No scientific principle is permanent. All principles are actually hypotheses for which enough evidences are provided to establish them. But this does not undermine the fact that a particular principle can never be improvised. In fact, history shows us that many scientific principles have been either disproved or improvised based on new evidences. The most glaring example of this is the different theories put forward about structure of atom, and still the most acceptable theory is considered true. Thus, it becomes extremely important that we should communicate science to our younger generation in such a way that they are not prejudiced about any established fact, and they should have enough knowledge and courage to put forward new questions.

Many times I have witnessed that students try to do an experiment by the steps given in the textbook, but they do not get results. Due to hesitation about the failure, they do not discuss their problem with the teacher and go ahead with the rote learning of the experiment because they have to pass the pen-paper examination. I would like to present the following instances to highlight this problem.

1.1 Instance 1

This incident was of sixth standard students where they were trying to prove that a salt solution conducts electricity. They tried many times by changing the solution, switch, electrodes, wires, etc., but could not succeed. When I came to know about this, I went and saw their experimental setup. I saw that the full setup was according to the given procedure in the book, but still they were not getting the bulb glow. I told them to keep on increasing the salt concentration and see when the bulb glows. They did that and found that at a certain concentration the bulb started glowing. The learning from this incident is even though the steps of the experiment is given in the textbook and the same can also be shown in a smart class, but the real experiment requires a guide who says them which are the critical steps to do an experiment. In this way, the students come to know when and where critical thinking is required.

1.2 Instance 2

In another incident about a different experiment, the students were getting the result but one group was getting better result than the other. This case is of third standard students where they were making a water purification model using plastic bottles, gravels of different sizes and sand. The students were divided into many groups of 5 members each. Instructions were given as to how to make the model, and steps and diagram was given in the textbook. Next day all the groups brought their models and demonstrated that indeed the setup was purifying dirty water. But some models were giving cleaner water than other models. Students asked why this was so, even though all have used the same steps of model making. Then I told them to measure the thickness of different layers of gravel and sand, and see if that has an effect on the cleanliness of water. Thus, the students came to know about the critical aspects of the model. With these incidents and many others, I felt that in learning-by-doing method the students require some guidance at some point to understand the criticality of the experiment.

2 Hypothesis

In the present world of knowledge explosion, most of out teaching time is devoted towards loading the students with information. We rarely use research-based techniques in our teaching curriculum. My hypothesis is that if our teaching methodology is designed in such a way that students are allowed to observe and collect data with some necessary guidance from the teacher, they will come to know about the points where critical thinking is required; and from there onwards, they can themselves come to the required conclusion. They will also get enough confidence towards thinking on innovative ideas.

3 Methodology

This study was conducted in two stages. The first stage included a survey of fifth standard students on two questionnaires. Questionnaire 1 was to measure critical thinking behaviour of the students in classroom. Questionnaire 2 was to assess their performance on questions that required critical thinking. The second stage of the study included assessment of performance of B.Sc. third year zoology students on a new topic which was taught by traditional lecture method to control group and by a new guided method to experimental group.

3.1 Stage I

Two surveys were conducted on fifth standard students of M.E.S. Kishore Kendra School. The first survey consisted of questionnaire 1 which was to identify the tendency in students to think critically. This questionnaire was prepared in consultation with the Department of Psychology, MES Degree College of Arts, Commerce and Science. The second survey consisted of questionnaire 2 which was to measure scores of students in science subject. This questionnaire included those questions from their syllabus which had a very little probability to be answered by rote learning. Each question required critical thinking on a particular topic.

3.2 Stage II

Due to administrative reasons, I could not get a chance to take class in the fifth standard myself; therefore, I continued my study on B.Sc. third years students which were divided into two sections - one control group and another experimental group. The topic chosen was "Parental Care in Fishes" which was a new topic for both the groups. This was a single-blind study because the topic was unknown to both the groups and students did not know whether they were belonging to control group or experimental group. For the control group, the topic was taught by the regular lecture method. For the experimental group, the topic was taught with the help of an overhead projector and hardcopies of an observation table which was prepared by the teacher. In this method, the students were instructed to fill the observation table by carefully observing the pictures related to the topic. During the entire period, the teacher acted as a guide to instruct them about critical aspects of observation. At the end of both classes, a test was conducted to assess the effectiveness of both teaching methodologies.

4 Results

4.1 Stage I Findings

In the first survey with questionnaire 1, it was found that 3.7% of the total students got marks between 0 to 5 out of 20, 11.9% of the total students got marks between 6 to 10 out of 20, 67.9% of the total students got marks between 11 to 15 out of 20, and 16.5% of the total students got marks between 16 to 20 out of 20. Overall, 84.4% of the total students scored more than 50% marks. This result indicates that most of the students think reasonably critically in the class.

Table 1: Questionnaire 1 Scores For Measuring Critical Thinking Behaviour

Marks	Number of Students	Percentage
0 to 5 in Q1	4	3.7%
6 to 10 in Q1	13	11.9%
11 to 15 in Q1	74	67.9%
16 to 20 in Q1	18	16.5%

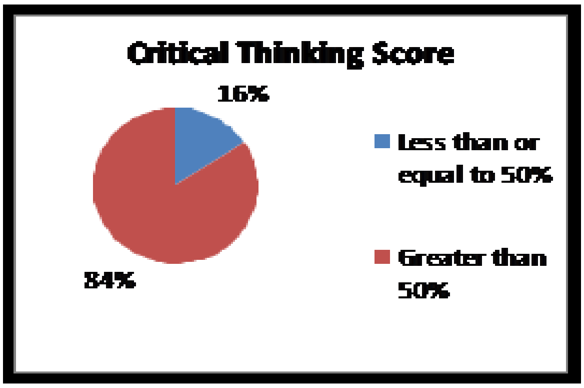

Figure 1

In the second survey with questionnaire 2, it was found that 0.9% of the total students scored between 0 to 5 out of 20, 10.1% of the total students scored between 6 to 10 out of 20, 18.4% of the total students scored between 11 to 14 out of 20, and 70.6% of the total students scored between 15 to 20 out of 20. Overall, 89% of the total students scored more than 50% marks.

Table 2: Questionnaire 2 Scores For Measuring Performance in Science

Marks	Number of Students	Percentage
0 to 5 in Q2	1	0.9%
6 to 10 in Q2	11	10.1%
11 to 15 in Q2	20	18.4%
16 to 20 in Q2	77	70.6%

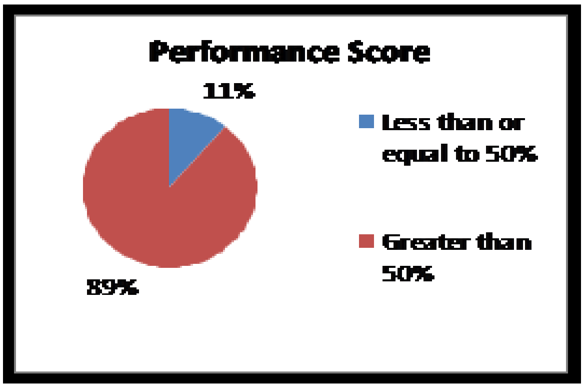

Figure 2

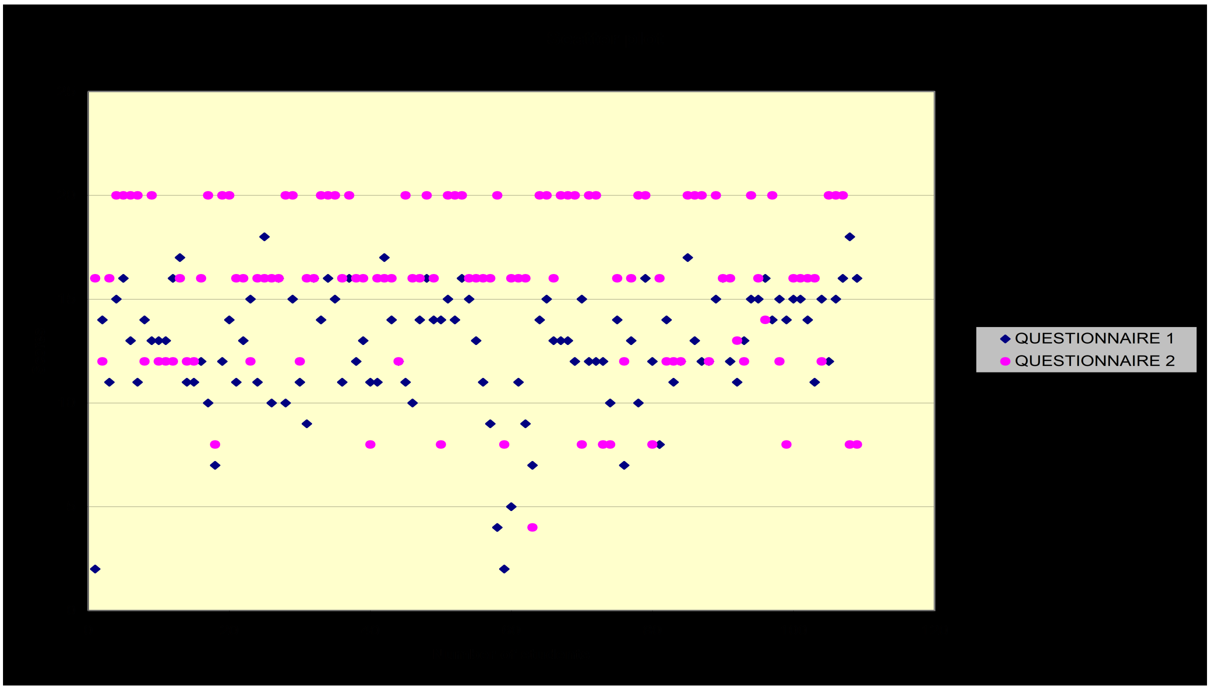

Figure 3: A Scatter Plot Showing Comparison of Scores of Both Questionnaire 1 and 2

4.2 Stage II Findings

In the control group, 0% of the total students scored between 0% to 24% marks, 8.5% of the total students scored between 25% to 49% marks, 70.2% of the total students scored between 50% to 74% marks, and 21.3% of the total students scored between 75% to 100% marks. Overall, 38.3% of the total students scored less than or equal to 50% marks and 61.7% of the students scored greater than 50% marks.

In the experimental group, 2.2% of the total students scored between 0% to 24% marks, 6.7% of the total students scored between 25% to 49% marks, 60.0% of the total students scored between 50% to 74% marks, and 31.1% of the total students scored between 75% to 100% marks. Overall, 24.4% of the total students scored less than or equal to 50% marks and 75.6% of the students scored greater than 50% marks.

Table 3: Category-wise Comparison of Control and Experimental Groups of B.Sc. Students

Category	Control group		Experimental group	
	Number of Students	Percentage	Number of Students	Percentage
0% to 24%	0	0.0%	1	2.2%
25% to 49%	4	8.5%	3	6.7%
50% to 74%	33	70.2%	27	60.0%
75% to 100%	10	21.3%	14	31.1%

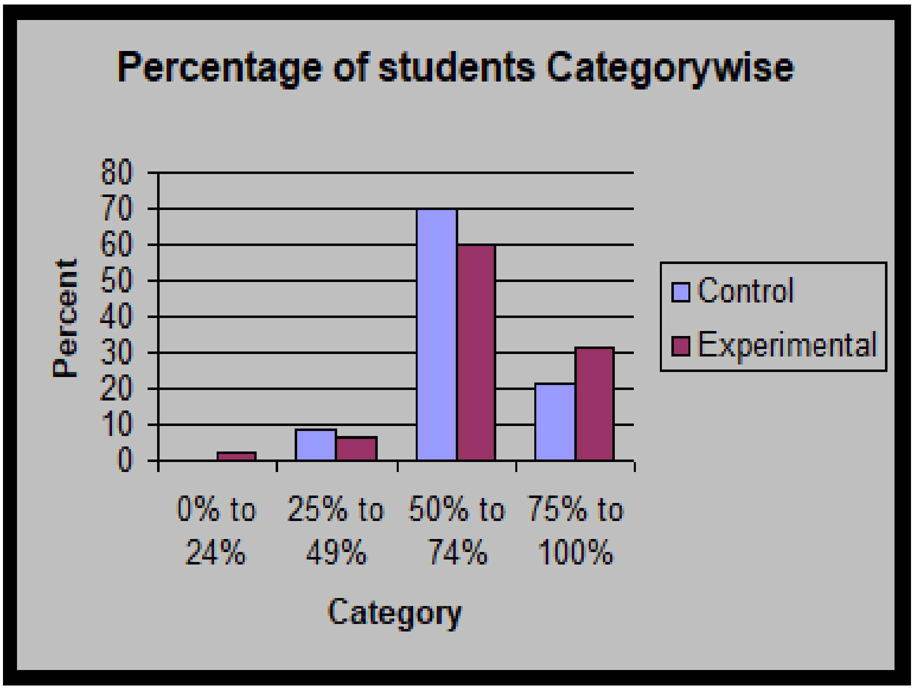

Figure 4

Table 4: Overall Comparison of Control and Experimental Groups Based on Marks Scored Less Than or Equal to 50% and More Than 50%

Category	Control (%)	Experimental (%)
Students scoring less than or equal to 50%	38.3	24.4
Students scoring greater than 50%	61.7	75.6

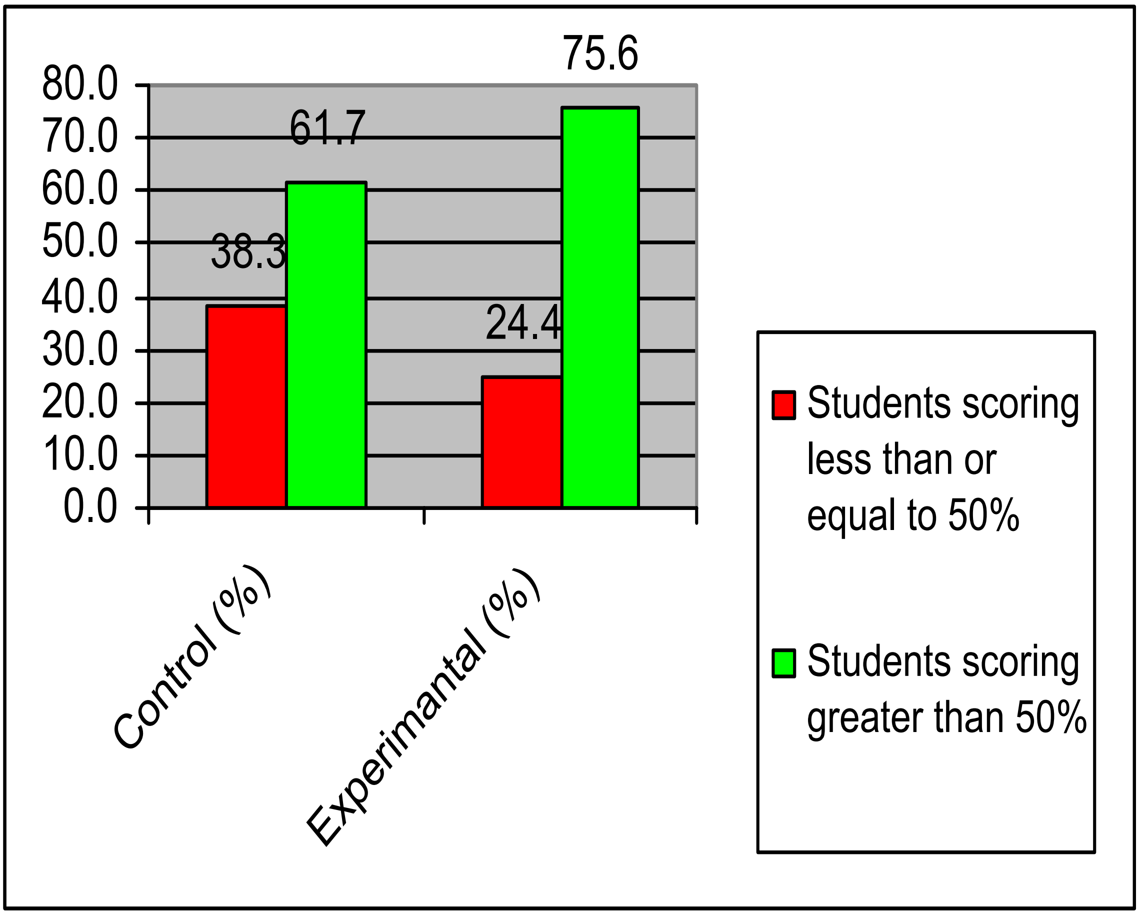

Figure 5

5 Discussion

Critical thinking is all about curiosity, flexibility, and keeping an open mind [1]. In another study, Anat Zohar and colleagues [2] tested 678 students of seventh grade and found that students with critical thinking training, along with biology, showed greater improvement in their analytical skills and not just for biology problems. The students trained in critical thinking also did a better job in solving everyday problems [2]. Philip Abrami and colleagues [11] studied 117 instances about teaching critical thinking and found that teaching young students by giving explicit instructions about different ways to reason and solve problems resulted in considerable improvement, even in young students [11]. These studies suggest that if methods of science teaching are such that they induce critical thinking, where stress is put on providing environment to the students to collect data, interpret and find conclusion on their own, it results in improved understanding of the scientific concepts.

In the present study, first of all a survey was conducted on 109 students of fifth grade to identify signs of critical thinking during classroom teaching (questionnaire 1). There were five questions with four multiple choice answers. Each question was assigned four marks. Each choice of each question was assigned marks from 1 to 4. The choices were based on the extent of critical thinking behaviour, where 1 mark was given to the least critical thinking and 4 marks were given to the most critical thinking. The highest marks that could be scored in questionnaire 1 was 20 and the lowest marks was 5. If a student scored 20, then it indicates that the student is highly critical about the subject and thinks independently about the topic. If a student scored 5, then it indicates that the student is not able to think independently and is dependent on the teacher for understanding the concepts. The next survey of the same 109 students of fifth grade was to find out the relationship between critical thinking behaviour and performance in a test which included questions that required critical thinking. These questions were not memory-based and there was a little chance that a student may score high by simply rote learning. This survey was accomplished using questionnaire 2.

The two surveys of fifth standard with a total sample size of 109 students indicate that most of the students think reasonably critically and they show good performance. This also indicates that there might be some direct correlation between critical thinkers and good performers. I then calculated the correlation coefficient by the Spearman's rank method and got a correlation coefficient value of +0.3, which means that there is a low positive correlation between critical thinkers and good performers.

But the most notable finding in these surveys was that 67.9% of the students scored between 50% to 75% marks in questionnaire 1, and 70.6% of the students scored between 75% to 100% marks in questionnaire 2. A scatter plot of the both questionnaires' marks shows that most of the students have reasonable critical thinking capacity. These data and the correlation coefficient together indicate that if a reasonably critically thinking student is given proper guidance during teaching, then it may result in a good improvement in understanding of the scientific topics in majority of the students.

Due to administrative reasons, I could not continue the second stage of my study in fifth standard. Therefore, I extrapolated the conclusion of the above two surveys, and assuming that the inherent nature of critical thinking does not change in a mature student, I conducted the second stage of my study in B.Sc. Zoology third year students of MES Degree College of Arts, Commerce and Science. The second stage of the study was designed in the following way. A total of 92 students of third year B.Sc. Zoology were divided into two groups. Group 1 consisted of 47 students and group 2 consisted of 45 students. Group 1 was taken as control group and group 2 was taken as experimental group. This was a single-blind study in which students did not know to which group they were belonging to, and the topic chosen for the class was also totally new to both the groups. The topic chosen was titled "Parental Care in Fishes." The chosen topic was first taught to the control group using traditional lecture method where the necessary diagrams were manually drawn on the board by the teacher. The students just listened to the lecture and noted down as many relevant points as they could. Immediately after the lecture, a test was conducted which included six questions, each having 4 multiple choice answers. Each question was assigned 2 marks, with total marks of 12. The nature of the question was such that they could not be answered simply by rote learning. Each question required critical thinking and analysis of the chosen topic.

The same topic was taught next to the experimental group. Here, the traditional lecture method was not used. Instead an overhead projector was used as a teaching aid and a number of pictures regarding the topic were shown to the students to simulate a kind of field trip for the students. In addition to this an observation table designed by the teacher was given to each student in which instructions were given to help the students to fill the data. Here the teacher acted as a guide for the student to help them in getting the data and filling the observation table. At the end of this session, the students were allowed to draw their own conclusions. Immediately after that, a test was conducted which included the same questions that were given to the control group. The findings of both groups were recorded.

On comparing the results of both the control and experimental groups, we find that there is an increase of 2.2% in the 0% to 24% category, a decrease of 1.8% in the 25% to 49% category, a decrease of 10.2% in the 50% to 74% category, and an increase of 9.8% in the 75% to 100% category. These results clearly indicate that there has been a considerable increase in the percentage of students scoring more than 75% marks. Also, there is a minor increase in the percentage of students scoring below 25% marks.

6 Conclusion

Based on the above results, it can be concluded that guided teaching can prove to be an improvised methodology of science teaching in schools as well as colleges, which can improve the performance of students by inducing critical thinking. In this method, the role of teacher is pivotal because the success of this method depends on the design of the lesson plan and selection of teaching aids. It is worth noting that the slight increase in the 0% to 24% category may be due to lack of interest in a new teaching methodology or due to lack of adjustability of the student to the new method. This may be possible because some students may feel more comfortable with an already established method of teaching. An improvised method of teaching may take some time to be well accepted by the teachers as well as students because it requires both of them to come out of their comfort zone and work together towards achieving better understanding of science. Apart from increasing the performance score, guided teaching is also likely to increase curiosity among students and inclination towards research.

I also would like to bring the attention of the reader towards the fact that in our country, it is still the pen-paper examination which gets the highest priority by both the teachers as well as students. This situation may act like a hindrance for innovative teaching methodologies to become well accepted in the teaching fraternity. Therefore, it is imperative that we should think about new strategies for conducting examinations which can assess both knowledge and research aptitude of the students. Additionally, I strongly believe that innovative methods of teaching and assessment will help to a large extent in popularizing science in our country.

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Applications of Singular Value Decomposition

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Abstract: Singular Value Decomposition is a general way of factorizing any real matrix of any size, not necessarily a square one. It is closely associated with the eigenvalues- eigenvectors factorization of a symmetric matrix.[3, 1]. In this paper we take a look at a few advantages of Singular Value Decomposition.

Keywords: Singular Value Decomposition, Psudeoinverse, Least squares solution, Least squares line, Least squares parabola, Least square error.

1 Introduction

Let A be a matrix of order r having m rows and n columns. The singular value decomposition of A is the factorization of A into the product of three matrices $A = U\Sigma V^T$ where

- $U(m$ by $m)$ is an orthogonal matrix where the columns of U, called left singular vectors of A, are eigenvectors of AA^T.
- $V(n$ by $n)$ is an orthogonal matrix where the columns of V, called right singular values of A, are eigenvectors of A^TA.
- $\Sigma(m$ by $n)$ is a diagonal matrix with r singular values (which are the square roots of the non-zero eigenvalues of both AA^T and A^TA) on its principal diagonal such that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m$ and zeroes elsewhere.

The Singular Value Decomposition or the "SVD" is the backbone of linear algebra. It is critical in fundamental areas such as image compresssion, web searching, signal processing, cryptography, pattern recognition, principal component analysis, control theory and many more [5, 6]. We now look at a few of them.

2 Pseudoinverse

If A is $m \times m$ non-singular matrix then there exists A^{-1} such that $AA^{-1} = A^{-1}A = I$. But if A is an $m \times n$ matrix, then we define the inverse called as Pseduoinverse. Thus pseudoinverse of a matrix A is a generalization of the inverse matrix denoted by A^+. The pseudoinverse is also called Moore-Penrose inverse, after the mathematicians Eliakim H. Moore and Roger Penrose.

For $A \in \mathbb{R}^{m\times n}$, the pseudoinverse of A is defined as $A^+ \in \mathbb{R}^{n\times m}$ satisfying all of the following four Penrose criteria, known as the Moore-Penrose conditions [3, 4].

1. $AA^+A = A$
2. $A^+AA^+ = A^+$
3. $(AA^+)^T = AA^+$
4. $(A^+A)^T = A$

$\therefore$ A^+ exists for any matrix with real entries and is unique.

2.1 Method of Solution

To determine the pseudoinverse of a given matrix, follow these steps

1. Find SVD for the given matrix i.e. $A = U\Sigma V^T$.
   (a) Compute the eigen values and corresponding eigen vectors for $B = AA^T$ and $C = A^TA$.
   (b) Let U be the matrix consisting of all the normalized eigen vectors corresponding to B and V be the matrix consisting of all the normalized eigen vectors corresponding to C. Let Σ be a matrix comprising of the eigen values (in the decreasing order).
   (c) Now $U\Sigma V^T$ should give A.
2. Using the formula $A^+ = V\Sigma^+U^T$, we find pseudoinverse of A [1, 7, 8].

2.2 Examples and Discussion

We consider a few examples to check the above explained methods.

Example 1: Determine the pseudoinverse of $\begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$.

Solution: Let $A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$. Finding SVD for the given A gives, $U = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$,
$V = \begin{bmatrix} \frac{-2}{\sqrt{14}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{21}} \\ \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{21}} \\ \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{6}} & \frac{-4}{\sqrt{21}} \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{7} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The pseudoinverse of A is given by $A^+ = V\Sigma^+U^T = \frac{1}{21}\begin{bmatrix} 10 & 4 \\ -1 & 8 \\ 2 & 5 \end{bmatrix}$.

Example 2: Determine the pseudoinverse of $\begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}$.

Solution: Let $A = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}$. Finding SVD for the given A yields, $U = \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & -\frac{5}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{2}{\sqrt{30}} \end{bmatrix}$,
$V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{12} & 0 \\ 0 & \sqrt{10} \end{bmatrix}$

The pseudoinverse of A is given by $A^+ = V\Sigma^+U^T = \begin{bmatrix} \frac{17}{60} & \frac{1}{15} & \frac{1}{6} \\ -\frac{1}{30} & \frac{4}{15} & \frac{1}{12} \end{bmatrix}$

Example 3: Determine the pseudoinverse of $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

Solution: Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Finding SVD for A we have, $U = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$, $V = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

and $\Sigma = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$

The pseudoinverse of A is given by $A^+ = V\Sigma^+U^T = \frac{1}{25}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Example 4: Determine the pseudoinverse of $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$.

Solution: Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$. Finding SVD for A we obtain, $U = \begin{bmatrix} 1 \end{bmatrix}, \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and

$$V = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

The pseudoinverse of A is given by $A^+ = V\Sigma^+U^T = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$

2.3 Applications of pseudoinverse

The pseudoinverse of a matrix has applications in various fields, to name a few:

1. It is employed in areas of data analysis and Principal Component Analysis (PCA).
2. It is an important factor in Image compression and digital image restoration.
3. Most importantly, it is used in finding least squares solution for a given over-determined system of equations.

3 Least Squares Solution

Consider a system of n equations in n variables say $Ax = y$, where A is invertible, then A has a unique solution. However, if $Ax = y$ is a system of n equations in m variables (where $n > m$) the system is said to be over-determined and generally does not have a solution. In such cases A is not a square matrix and thus A^{-1} does not exist. With the help of pseudoinverse A^+, we find a solution called **least-squares solution**, for the over determined system. This is not a true solution, but in some sense the closest we can get to a true solution for the system. [1, 9]

Analyzing data to interpret and predict events is common to business, engineering, physical and social sciences. If such data is plotted, they constitute a scatter diagram, which may provide insight into the underlying relationship between such variables. In order to find an equation suitable to express this relationship, we can make use of an equation of a line or curve that in some sense fits "best" for the obtained data. One criteria that has been found most satisfactory is called **the least squares line or curve**, which is found by solving an over-determined system of equation as mentioned earlier. In both the situations the best result is the one obtained by minimizing certain squares, thus rendering the name least squares. [2, 5]

Let $Ax = b$ be an inconsistent system of equations with m equations and n variables where $(m > n)$. For most choices of b, there exists no $n-$ vector x satisfying $Ax = b$. Thus, we seek an alternate x for which $r = Ax - b$, which is called the residual (for the equation $Ax = b$), is small as possible. i.e, choose x such that $\|Ax = b\|$ small, then we have $Ax \approx b$. When a least squares solution $\hat{x}$ makes $Ax \approx b$, the distance from b to $A\hat{x}$ is called the **least-squares error** of this approximation. [3]

3.1 Method of Solution

3.1.1 To determine the Least Squares Solution for a given over-determined system of equations

1. Express the given system of equations in the form $Ax = b$ where A forms the matrix of coefficients and b the matrix of column vectors.
2. Consider $C = A^T A$, evaluate C^{-1} and $(A^T b)$.
3. To solve for x, compute $x = C^{-1}A^T b$. The solution so obtained will be the least squares solution to given system of equations.

3.1.2 To find the Least Squares Line for the given data points

Let $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ be the points in the data, which lie approximately on a line. To find the equation of this line which best fits these points, follow these steps
1. The equation of the line is given by $y = a + bx$.
2. Substituting the points into the equation of the line, we obtain the following over-determined system of equations $$\begin{aligned} a + bx_1 &= y_1 \\ a + bx_2 &= y_2 \\ a + bx_3 &= y_3 \end{aligned}$$
3. Let A form the matrix of coefficients and y be column vectors, then we have $$A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \end{bmatrix} \text{ and } y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$
4. Find the least squares solution for the above system using $x = C^{-1}A^T y$
5. The equation of the least squares line for this data is $y = a' + b'x$.

3.1.3 To find the Least Squares Parabola for the given data points

Let $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ be the points in the data, which lie approximately on a parabola. To find the equation of this parabola which best fits these points, follow these:
1. Let the equation of the parabola is given by $y = a + bx + cx^2$.
2. Substituting the points into the equation of the parabola, we obtain the following over-determined system of equations $$\begin{aligned} a + bx_1 + cx_1^2 &= y_1 \\ a + bx_2 + cx_2^2 &= y_2 \\ a + bx_3 + cx_3^2 &= y_3 \\ a + bx_4 + cx_4^2 &= y_4 \end{aligned}$$
3. Let A form the matrix of coefficients and y be column vectors, then we have $$A = \begin{bmatrix} 1 & x_1 & {x_1}^2 \\ 1 & x_2 & {x_2}^2 \\ 1 & x_3 & {x_3}^2 \\ 1 & x_4 & {x_4}^2 \end{bmatrix} \text{ and } y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

4. Find the least squares solution for the above system using $x = C^{-1}A^Ty$.
5. The equation of the least squares parabola for this data is $y = a' + b'x + c'x^2$.

3.1.4 To determine the Least Squares Solution Error associated with the Least Squares Solution for a given system of over-determined equations

1. Let A and b be the given rectangular and column matrices respectively.
2. Find the least square solution.
3. Compute $Ax - b$.
4. The least squares error is given by $\|Ax - b\|$.

3.2 Examples and Discussion

We consider a few examples to understand the above explained methods.

Example 5: Determine the least squares solution of the over determined set of equations
$-x + 2y = 4$
$2x - 3y = 1$
$-x + 3y = 2$

Solution: Let $A = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

Consider $C = A^TA = \begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix}$ and $A^Tb = \begin{bmatrix} -4 \\ 11 \end{bmatrix}$

The least squares solution is given by $\hat{x} = C^{-1}A^Tb = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Example 6: Describe all the possible least squares solution of $Ax = b$ for the following
$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 7 \\ 2 \\ 3 \\ 6 \\ 5 \\ 4 \end{bmatrix}$

Solution: Compute $A^TA = \begin{bmatrix} 6 & 3 & 3 \\ 3 & 3 & 0 \\ 3 & 0 & 3 \end{bmatrix}$ and $A^Tb = \begin{bmatrix} 27 \\ 12 \\ 15 \end{bmatrix}$

The Augmented matrix for $A^TAx = A^Tb$ is given by $[A^TA : A^Tb] = \begin{bmatrix} 6 & 3 & 3 & : & 27 \\ 3 & 3 & 0 & : & 12 \\ 3 & 0 & 3 & : & 15 \end{bmatrix}$

Applying row operations, we get $[A^TA : A^Tb] \approx \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The general solution is $x_1 = 5 - x_3, x_2 = -1 + x_3$.

So the general least squares solution of $Ax = b$ has the form $\hat{x} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} x_3$

Example 7: Determine the least squares line for the data points $(1, 1), (2, 3), (3, 7)$.

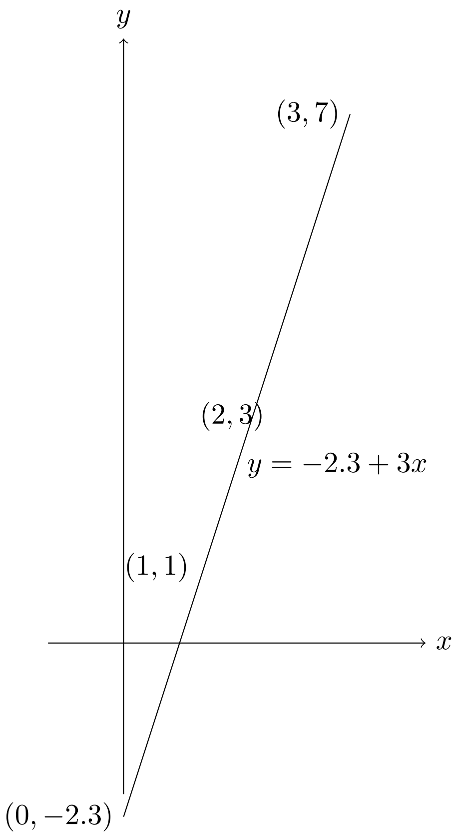
Solution: Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ and and $y = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$

Consider $C = A^T A = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \Rightarrow C^{-1} = \frac{1}{6} \begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix}$

The least squares solution is $x = \begin{bmatrix} -2.3 \\ 3 \end{bmatrix}$

The equation of the least squares line for the given data is $y = -2.3 + 3x$.

i.e., this line is generally thought to be the line of best fit for the given points.



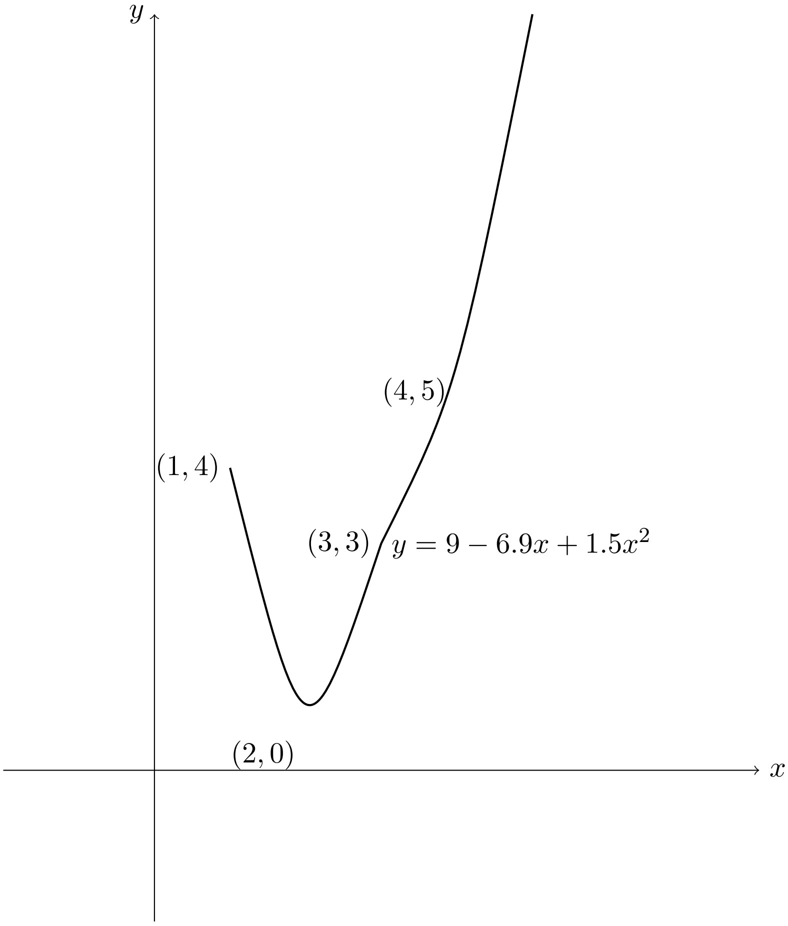
Example 8: Determine the least squares parabola for the points $(1, 4), (2, 0), (3, 3), (4, 5)$.

Solution: Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}$ and $y = \begin{bmatrix} 4 \\ 0 \\ 3 \\ 5 \end{bmatrix}$

Consider $C = \begin{bmatrix} 4 & 10 & 30 \\ 10 & 30 & 100 \\ 30 & 100 & 354 \end{bmatrix} \implies C^{-1} = \frac{1}{80} \begin{bmatrix} 620 & -540 & 100 \\ -540 & 516 & -100 \\ 100 & -100 & 20 \end{bmatrix}$

The least squares solution is given by $x = \begin{bmatrix} 9 \\ -6.9 \\ 1.5 \end{bmatrix}$

The equation of the least squares parabola for the given data is $y = 9 - 6.9x + 1.5x^2$.



Example 9: The rate of gasoline consumption of a car depends upon its speed. To determine the effect of speed on fuel consumption, the Federal Highway administration tested cars of various weights a various speeds. One car weighing $3,980$ pounds have the following results. Determine the least squares line corresponding to the data. Use this equation to predict the mileage per gallon of this car at 55 mph. (Speed is in miles per hour and gas used in miles per gallon)

Speed	30	40	50	60	70
Gas used	18.25	20	16.32	15.77	13.61

Solution: Consider the equation $G = a + bM$, where G is the rate of Gasoline consumption and M is the mileage per gallon.

Constitute $A = \begin{bmatrix} 1 & 30 \\ 1 & 40 \\ 1 & 50 \\ 1 & 60 \\ 1 & 70 \end{bmatrix}$ and $y = \begin{bmatrix} 18.25 \\ 20 \\ 16.32 \\ 15.77 \\ 13.61 \end{bmatrix}$

Consider $C = A^T A = \begin{bmatrix} 13500 & 250 \\ 250 & 5 \end{bmatrix}$

Finding the least squares solution to the above data, we get $x = \begin{bmatrix} 23.545 \\ -0.1351 \end{bmatrix}$

The least squares line will be $G = 23.545 - 0.1351M$

Thus the fuel consumption of the car at 55 mph will be $G = 16.1145$ miles per gallon.

Example 10: An advertising company has arrived at the following statistics relating the amount of money spend an advertising a certain product to be realized sale of the product. Find the least squares line and use it to predict the sales when $5,000$ is spend on advertising

Dollars spend	1000	1500	2000	2500	3000	3500
Units sold	520	540	582	600	610	615

Solution: Consider the equation $U = a + bD$, where D is the amount of dollars spent and U is the units.

Constitute $A = \begin{bmatrix} 1 & 1000 \\ 1 & 1500 \\ 1 & 2000 \\ 1 & 2500 \\ 1 & 3000 \\ 1 & 3500 \end{bmatrix}$ and and $y = \begin{bmatrix} 520 \\ 540 \\ 582 \\ 600 \\ 610 \\ 615 \end{bmatrix}$

Consider $C = A^T A = \begin{bmatrix} 6 & 13500 \\ 13500 & 34750000 \end{bmatrix}$

Finding the least squares solution we obtain $x = \begin{bmatrix} 487.4476 \\ 0.040171428 \end{bmatrix}$

The least squares line will be $U = 487.4476 + 0.040171428D$.
Thus the number of products sold when \$5000 is spent on advertising will be $U = 688$ units.

Example 11: Compute the least-square error associated with the least squares solution for the system of equation

$$\begin{aligned} x_1 - 2x_2 &= 3 \\ -x_1 + 2x_2 &= 1 \\ 3x_2 &= -4 \\ 2x_1 + 5x_2 &= 2 \end{aligned}$$

Solution: Let $A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix}$

The least squares solution is $x = C^{-1}A^T b = \begin{bmatrix} 4/3 \\ -1/3 \end{bmatrix}$

Compute $Ax - b = \begin{bmatrix} -1 \\ -3 \\ 3 \\ -1 \end{bmatrix}$

The Least-squares error is given by = $\|Ax - b\| = 2\sqrt{5}$.

Example 12: Let $A = \begin{bmatrix} 3 & 4 \\ -2 & 1 \\ 3 & 4 \end{bmatrix}$, $b = \begin{bmatrix} 11 \\ -9 \\ 5 \end{bmatrix}$, $u = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$. Compute Au and Av and compare them with b. Is u a least-squares solution of $Ax = b$? (Answer this without computing the least squares solution) [2].

Solution: Compute $b - Au = \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix} \implies \|b - Au\| = \sqrt{40}$

Compute $b - Av = \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix} \implies \|b - Av\| = \sqrt{29}$

Here Av is more close to b than Au is closer to b.
Thus u cannot be a least squares solution of $Ax = b$.

3.3 Applications of Least Squares Solution

The least squares solution is used to find the best fit of the polynomial for the given data and also in signal denoising and image deblurring.

4 Conclusion

In this paper we have just given an overview of the vast advantages of Singular Value Decomposition. SVD is very apt for numerically stable computations. We have effectively shown the basics of the pseudoinverse, from where it is derived, how to calculate it and also its applications. The most important application of pseudoinverse is in data fitting. The best fit in the least-squares sense minimizes the sum of squared residuals. The graphical method has the principle of least squares which provides a unique set of values to the constants and hence suggests a curve of best fit for the given data.

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Applications of Matrices to Economics and Demography

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Abstract: This article consists of some examples of real world problems which can be solved using Mathematical Modelling technique. In this article we have discussed about Leontief Input and Output Model in Economics and Markov Chains in Demography.

Keywords: Mathematical Modelling, Leontief Input and Output Model, Markov Chains, Demography.

1 Introduction

Mathematical Modelling is a technique which essentially consists of translating real world problems into mathematical problems, solving the mathematical problems and interpreting these solutions in the language of the real world. In other words, mathematical modelling is a method of simulating real life situations with mathematical equations to forecast their future behaviour [1]. Mathematical Modelling of large-scale systems presents its own special problems. However mathematical modellers from all disciplines - mathematics, statistics, computer science, physics, engineering, social sciences - are meeting the challenges with courage [3].

2 Leontief Input and Output Model in Economics

This topic introduces an important application of matrix inversion in modern economic theory. The leontief model describes a simplified view of an economy. Its goal is to predict the proper level of goods or services.

Wassily Leontief received a Nobel prize in 1973 for his work in this field. His model is a basis for more models currently being used in many parts of the World. This model can be applied to any size economy from a small business to the whole world. The main goal of the Leontief input-output model is to balance the total amount of goods produced to the total demand for that production,
i.e., Amount produced (X) = intermediate demand + final demand.

2.1 Gauss-Jordan method for finding the inverse of a matrix

To find the inverse of a $n \times n$ square matrix through sequence of Row elementary row operations here we use Gauss-Jordan method. Which will reduce A to I_n. It turns out that the same sequence of row operations will reduce I_n to A^{-1} [3]. This method involves the following steps.

- The given matrix A is written as augmented matrix $[A : I]$.

- An elementary row operation on an $n \times n$ augmented matrix.
- Through Gauss-Jordan reduction on the augmented matrix we have reduce the A as I. i.e., $[I : A^{-1}]$ matrix, which gives us the inverse of A.
- If we cannot reduce A to I using row operations, then A has no inverse.

2.2 Applications of Leontief Input and Output Model

There are mainly two applications of Leontief model they are Open model and Closed model. Open model finds the amount of production needed to satisfy an increase in demand whereas the closed model deals only with the income of each industry. The closed model means that all inputs into production are produced and inputs exist merely to serve as input.

1. This method is most commonly used for estimating the impacts of positive or negative economic shocks and analyzing the ripple effects throughout an economy.
2. It is used to Forecasting and Estimate the impact of investment.
3. Indirect cost allocation and Accurate cost detection.
4. Now a days many people apply the input-output methodology to empirical problems requiring economic analysis.
5. The real strength of the input-output methodology lay in its practical uses as an impliment of economic analysis.
6. Households produces these factor inputs using commodities. As a matter of fact the Leontief open production Model provides us with a powerful economic analysis tool in the form of input-output analysis.

3 Markov Chains

Markov Chain are an important mathematical tool in Stochastic process. The underlying idea is the Markov property, in other words, some predictions about stochastic processes can be simplified by viewing the future as independent of past, given the present state of the process. This is used to simplify predictions about the future state of a stochastic process [1].

Markov chain is a stochastic process that satisfies the Markov property, which means that the past and feature are independent when the present is known, this means that if one known the current state of the process, then no additional information of it's past state is required to make the possible predictions of it's future [4]. The transition matrix P of a Markov chain is said to be regular if for some power of P all the components are positive , the chain is then called a regular Markov chain [1]. State of a Markov chain is an absorbing state if $P_{ii} = 1$.

A Markov chain is an absorbing chain if and only if the following two conditions are satisfied.

(i) The chain has at least one absorbing state.
(ii) It is possible to go from any non-absorbing state to an absorbing state [1].

3.1 Demography

Demography is the study of statistics such as births, deaths, income, or the incidence of disease, which illustrate the changing structure of human populations. Demographic analysis can cover whole societies or groups defined by criteria such as education, nationality, religion, and ethnicity. Educational institutions usually treat demography as a field of sociology, though there are a number of independent demography departments. Formal demography limits its object of study to the measurement of population processes, while the broader field of social demography or population studies also analyses the relationships between economic, social, cultural, and biological processes influencing a population [1].

3.2 Method of Solution

Let us explain this method by taking annular population distributions. This could be described by a sequence of vector X_0.

$$X_1{=}PX_0,\ \ X_2{=}PX_1,\ \ X_3{=}PX_2,\dots$$

P is a matrix of transition probabilities that takes it from one vector in the sequence to the following vectors. Such a sequence (or chain) of vectors is called a Markov chain. Here in Markov chains where the sequence $X_0, X_1, X_2, \dots$ converges to some fixed vector X, where

$$PX = X$$

The population movement will be in a "steady-state" with the total city population and total suburban population remaining constant.
$\therefore$ we can write $X_0, X_1, X_2, \dots \to X$
Since such a vector X satisfies $PX = X$, it would be an eigen vector of P corresponding to the eigenvalue 1.
By knowing of the existence and value of such a vector would gives information about the long term behavior of the population distribution using the below theorem [1, 4].

3.3 Applications of Markov Chains in Demography

1. **Signal Processing:** Due to the dynamic nature of stochastic Markov chain, it has become a powerful tool for sequential signal processing specially in wireless communication. Some recent works can be found in the special issue on Monte-Carlo method for statistical signal processing.
2. **Counting 0-1 tables:** Many ecology education and sociology problems involve with large 0-1 tables.In order to calculate the exact p-value for various text statistics on those tables, it is often required to count the total number of tables with certain constraints, such as fixed margin.
3. **Target tracking:** Designing sophisticated target tracking algorithm is an important task to both civilian and military surveillance system, particularly when a radar, sonar or optical sensor is operated in the present of clutter or when innovation are *non-Gaussian*. Using stochastic Markov chain for target tracking problems.

4 Examples and Discussions

Example 1: Consider an economy consisting of three industries having the following input-output matrix A. Determine the output levels required of the industries to meet the demands of the other industries to meet the demand of the other industries and of the open sector in each case [1], where $A = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{3}{10} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$, $D = \begin{bmatrix} 9 \\ 12 \\ 16 \end{bmatrix}$, $\begin{bmatrix} 6 \\ 9 \\ 8 \end{bmatrix}$ and $\begin{bmatrix} 12 \\ 18 \\ 32 \end{bmatrix}$, the units of D are millions of dollars.

Solution: We wish to compute the output levels X that corresponds to the various open sector demands D. X is given by the equation $X = AX + D$.

$$\text{Let } A = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{3}{10} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}, D = \begin{bmatrix} 9 \\ 12 \\ 16 \end{bmatrix}, \begin{bmatrix} 6 \\ 9 \\ 8 \end{bmatrix} \text{ and } \begin{bmatrix} 12 \\ 18 \\ 32 \end{bmatrix}$$

$$\text{We have, } X = (I - A)^{-1} D$$

$$\text{Hence, } (I - A)^{-1} = \frac{Adj(I - A)}{det(I - A)} = \begin{bmatrix} \frac{5}{3} & \frac{2}{3} & \frac{5}{8} \\ \frac{5}{3} & \frac{8}{3} & \frac{5}{8} \\ 0 & 0 & \frac{5}{4} \end{bmatrix} \text{ and } X = \begin{bmatrix} \frac{5}{3} & \frac{2}{3} & \frac{5}{8} \\ \frac{5}{3} & \frac{8}{3} & \frac{5}{8} \\ 0 & 0 & \frac{5}{4} \end{bmatrix}$$

$$\begin{bmatrix} 9 & 6 & 12 \\ 12 & 9 & 18 \\ 16 & 8 & 32 \end{bmatrix} = \begin{bmatrix} 33 & 21 & 52 \\ 57 & 39 & 88 \\ 20 & 10 & 40 \end{bmatrix}$$

The output levels necessary to meet the demands
$\begin{bmatrix} 9 \\ 12 \\ 16 \end{bmatrix}, \begin{bmatrix} 6 \\ 9 \\ 8 \end{bmatrix}$ and $\begin{bmatrix} 12 \\ 18 \\ 32 \end{bmatrix}$ are $\begin{bmatrix} 33 \\ 57 \\ 20 \end{bmatrix}, \begin{bmatrix} 21 \\ 39 \\ 10 \end{bmatrix}$ and $\begin{bmatrix} 52 \\ 88 \\ 40 \end{bmatrix}$ respectively.
(The units are millions of dollars)

Example 2: Determine the output levels required of each industry in each situation to meet the demands of the other industries and of the open sector.
$\begin{bmatrix} 0.20 & 0.20 & 0.10 \\ 0 & 0.40 & 0.20 \\ 0 & 0.20 & 0.60 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 8 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 8 \\ 16 \end{bmatrix}$ and $\begin{bmatrix} 8 \\ 24 \\ 8 \end{bmatrix}$

Solution:

$$\text{Let } (I - A) = \begin{bmatrix} 0.8 & -0.20 & 0.10 \\ 0 & 0.6 & -0.20 \\ 0 & -0.2 & 0.4 \end{bmatrix}$$

$$\text{Then } (I - A)^{-1} = \frac{Adj(I - A)}{det(I - A)}) = \begin{bmatrix} 1.25 & 0.625 & 0.625 \\ 0 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\text{We have, } X = (I - A)^{-1} D = \begin{bmatrix} 15 & 15 & 30 \\ 24 & 32 & 56 \\ 32 & 56 & 48 \end{bmatrix}.$$

The output levels necessary to meet the demands
$\begin{bmatrix} 4 \\ 8 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 8 \\ 16 \end{bmatrix}$ and $\begin{bmatrix} 8 \\ 24 \\ 8 \end{bmatrix}$ are $\begin{bmatrix} 15 \\ 24 \\ 32 \end{bmatrix}, \begin{bmatrix} 15 \\ 32 \\ 56 \end{bmatrix}$ and $\begin{bmatrix} 30 \\ 56 \\ 48 \end{bmatrix}$ respectively.

Example 3: Solve the Leontief production equation for an economy with three sectors given that,

$A = \begin{bmatrix} 0.2 & 0.2 & 0 \\ 0.3 & 0.1 & 0.3 \\ 0.1 & 0 & 0.2 \end{bmatrix}.$

Suppose the final demand is 40 units for manufacturing 60 units for Agriculture and 75 units for services. Find the production level X that will satisfy this demands.

Solution:

$$\text{Given } A = \begin{bmatrix} 0.2 & 0.2 & 0 \\ 0.3 & 0.1 & 0.3 \\ 0.1 & 0 & 0.2 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Then, } (I-A)^{-1} = \begin{bmatrix} 1.3793 & 0.30651 & 0.1149 \\ 0.45977 & 1.22605 & 0.45977 \\ 0.17241 & 0.03831 & 1.26436 \end{bmatrix}.$$

$$\text{We know that, } X = (I-A)^{-1}D = \begin{bmatrix} 1.3793 & 0.30651 & 0.1149 \\ 0.45977 & 1.22605 & 0.45977 \\ 0.17241 & 0.03831 & 1.26436 \end{bmatrix} \begin{bmatrix} 40 \\ 60 \\ 80 \end{bmatrix} = \begin{bmatrix} 82.754 \\ 128.73 \\ 110.343 \end{bmatrix}.$$

The production level will satisfy these demands, $\begin{bmatrix} 40 \\ 60 \\ 80 \end{bmatrix}, \begin{bmatrix} 82.754 \\ 128.73 \\ 110.343 \end{bmatrix}$ respectively.

Example 4: Construct a model of population flow between cities, suburbs and non-metropolitan areas of the united state. There respective population in 2000 are 58 million, 142 million, and 60 million. The stochastic matrix given by the probabilities of the moves is

$$\begin{array}{c c} & \begin{array}{ccc} \text{cities} & \text{suburbans} & \text{non-metro} \end{array} \\ \begin{array}{c} \text{cities} \\ \text{suburbans} \\ \text{non-metro} \end{array} & \begin{bmatrix} 0.96 & 0.01 & 0.015 \\ 0.03 & 0.98 & 0.005 \\ 0.01 & 0.01 & 0.98 \end{bmatrix} \end{array}$$

Predict the population of city, suburban and non-metropolitan ares for 2001 and 2002. If a person is living in the city in 2000. What is the probability that the person will be living in a non-metropolitan area in 2002?

Solution:

$$\text{The given transition matrix is } P = \begin{array}{c c} \begin{array}{ccc} \text{cities} & \text{suburbans} & \text{non-metro} \end{array} & \\ \begin{bmatrix} 0.96 & 0.01 & 0.015 \\ 0.03 & 0.98 & 0.005 \\ 0.01 & 0.01 & 0.98 \end{bmatrix} & \begin{array}{c} \text{cities} \\ \text{suburbans} \\ \text{non-metro} \end{array} \end{array}$$

Let the population of U.S in 2000 can be written as,

$$X_0 = \begin{bmatrix} 58 \\ 142 \\ 60 \end{bmatrix} \begin{array}{c} \text{cities} \\ \text{suburbans} \\ \text{non-metro} \end{array}$$

We know that, $X_n = PX_{n-1}$

If 2000 is the base year. Let X_0 be the population is 2000. Then population distribution for the years 2001 and 2002 are,

$$X_1 = PX_0 = \begin{bmatrix} 58.32 \\ 140.96 \\ 60.72 \end{bmatrix} \text{ in 2001,}$$

$$X_2 = PX_1 = \begin{bmatrix} 58.65 \\ 139.97 \\ 61.41 \end{bmatrix} \text{ in 2002.}$$

Let P_2 and P_3 be the probabilities of a person living in non-metropolitan area in 2001 and 2002 respectively.

$$P^2 = PP = \begin{bmatrix} 0.922 & 0.195 & 0.029 \\ 0.05 & 0.960 & 0.010 \\ 0.019 & 0.019 & 0.960 \end{bmatrix} \text{ in 2001,}$$

$$P^3 = P^2P = \begin{bmatrix} 0.884 & 0.281 & 0.0423 \\ 0.076 & 0.941 & 0.0153 \\ 0.028 & 0.028 & 0.941 \end{bmatrix} \text{ in 2002.}$$

Hence, the probability of a person living in non-metro area is 0.0423 in 2002.

Example 5: The population of U.S metropolitan and non-metropolitan areas in 2000 are described by the following vectors X_0 (in units of one million) and the population in the following years are given by a Markov chain with the transition matrix P.

$$X_0 = \begin{bmatrix} 200 \\ 60 \end{bmatrix} \begin{array}{c} \text{metro} \\ \text{non metro} \end{array}$$

$$P = \begin{array}{c} \begin{array}{cc} \text{metro} & \text{non-metro} \end{array} \\ \begin{bmatrix} 0.99 & 0.02 \\ 0.01 & 0.98 \end{bmatrix} \end{array} \begin{array}{c} \\ \text{metro} \\ \text{non-metro} \end{array}$$

Determine the long-term predictions for metro and non-metro populations, assuming no change in their population.

Solution:

$$\text{Given } P = \begin{array}{c} \begin{array}{cc} \text{metro} & \text{non-metro} \end{array} \\ \begin{bmatrix} 0.99 & 0.02 \\ 0.01 & 0.98 \end{bmatrix} \end{array} \begin{array}{c} \\ \text{metro} \\ \text{non-metro} \end{array}$$

$$X_0 = \begin{bmatrix} 200 \\ 60 \end{bmatrix} \begin{array}{c} \text{metro} \\ \text{non metro} \end{array}$$

Since all the elements of P are positive. Hence the chain is regular. Since P is regular,

$$PX = X$$
$$(P - I)X = 0$$
$$\begin{bmatrix} 0.99 - 1 & 0.02 \\ 0.01 & 0.98 - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$-0.01x_1 + 0.02x_2 = 0$$
$$0.01x_1 - 0.02x_2 = 0$$
$$x_1 = 2x_2$$

Choose, $x_2 = K$
Hence $x_1 = 2K$
where K is a scalar.
The eigen vector of P corresponding to $\lambda = 1$ are non-zero vectors of the form

$$X = K \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

where X is a steady-state vector.
Given that there is is no total annual population change over the year.

$$X = X_0$$
$$2K + K = 200 + 60$$
$$K = 86.66$$

The study state vector is, $X = \begin{bmatrix} 173.32 \\ 86.66 \end{bmatrix}$
Hence the long term prediction in U.S. city population is 173 and U.S. suburban population is 87.
Let $Q = \begin{bmatrix} 2s & 2s \\ s & s \end{bmatrix}$, where Q is stochastic matrix.

$$2s + s = 1$$
$$s = 0.33$$
$$Q = \begin{bmatrix} 0.66 & 0.66 \\ 0.33 & 0.33 \end{bmatrix}$$

$$P^2 = PP = \begin{bmatrix} 0.99 & 0.02 \\ 0.01 & 0.98 \end{bmatrix} \begin{bmatrix} 0.99 & 0.02 \\ 0.01 & 0.98 \end{bmatrix} = \begin{bmatrix} 0.98 & 0.03 \\ 0.01 & 0.96 \end{bmatrix}$$

$$P^3 = P^2P = \begin{bmatrix} 0.98 & 0.03 \\ 0.01 & 0.96 \end{bmatrix} \begin{bmatrix} 0.99 & 0.02 \\ 0.01 & 0.98 \end{bmatrix} = \begin{bmatrix} 0.97 & 0.04 \\ 0.01 & 0.94 \end{bmatrix}$$

$$P^4 = P^2P^2 = \begin{bmatrix} 0.98 & 0.03 \\ 0.01 & 0.96 \end{bmatrix} \begin{bmatrix} 0.98 & 0.03 \\ 0.01 & 0.96 \end{bmatrix} = \begin{bmatrix} 0.96 & 0.05 \\ 0.019 & 0.92 \end{bmatrix}$$

$$P^5 = P^3P^2 = \begin{bmatrix} 0.97 & 0.04 \\ 0.01 & 0.94 \end{bmatrix} \begin{bmatrix} 0.98 & 0.03 \\ 0.01 & 0.96 \end{bmatrix} = \begin{bmatrix} 0.9546 & 0.0675 \\ 0.0192 & 0.9027 \end{bmatrix}$$

$$P^6 = P^4P^2 = \begin{bmatrix} 0.96 & 0.05 \\ 0.019 & 0.92 \end{bmatrix} \begin{bmatrix} 0.98 & 0.03 \\ 0.01 & 0.96 \end{bmatrix} = \begin{bmatrix} 0.9413 & 0.0768 \\ 0.0278 & 0.8837 \end{bmatrix}$$

This is the long term prediction.

Example 6: Consider a genetic model in which the offspring of guinea pigs are crossed with hybrid only the transition matrix for that model is as following.

$$P = \begin{array}{c c} \begin{array}{ccc} AA & Aa & aa \end{array} & \\ \begin{bmatrix} 0.5 & 0.25 & 0 \\ 0.5 & 0.5 & 0.5 \\ 0 & 0.25 & 0.5 \end{bmatrix} & \begin{array}{c} AA \\ Aa \\ aa \end{array} \end{array}$$

where AA, Aa are pair of genes having long hair and aa are pair of genes having short hair. Prove that P is regular and determine Q. What information about the long term distribution of guinea pigs does it give?

Solution: First we prove that P is regular.

$$P^2 = \begin{bmatrix} 0.375 & 0.25 & 0.125 \\ 0.5 & 0.5 & 0.5 \\ 0.125 & 0.25 & 0.375 \end{bmatrix}.$$

Thus, P is regular.

$$P^3 = \begin{bmatrix} 0.312 & 0.25 & 0.1875 \\ 0.5 & 0.5 & 0.5 \\ 0.1875 & 0.25 & 0.3125 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} 0.2812 & 0.25 & 0.2187 \\ 0.5 & 0.5 & 0.5 \\ 0.2187 & 0.25 & 0.2812 \end{bmatrix}$$

$$P^5 = \begin{bmatrix} 0.2656 & 0.25 & 0.2343 \\ 0.5 & 0.5 & 0.5 \\ 0.2343 & 0.25 & 0.2656 \end{bmatrix}$$

$$P^6 = \begin{bmatrix} 0.2578 & 0.25 & 0.2426 \\ 0.5 & 0.5 & 0.5 \\ 0.2426 & 0.25 & 0.2578 \end{bmatrix}$$

$$P^7 = \begin{bmatrix} 0.253 & 0.25 & 0.246 \\ 0.5 & 0.5 & 0.5 \\ 0.246 & 0.25 & 0.253 \end{bmatrix}$$

We know that $P, P^2, P^3, \cdots, Q$,

$$\text{Hence } Q = \begin{bmatrix} 0.25 & 0.25 & 0.25 \\ 0.5 & 0.5 & 0.5 \\ 0.25 & 0.25 & 0.25 \end{bmatrix}$$

This is the long-term distribution.

5 Conclusion

In this article we consider some examples of real world problems like Leontief input and output model and Markov Chains. Here we use Mathematical Modelling technique to solve Leontief Input and Output Model in Economics and Markov Chains in Demography.

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