MES BULLETIN OF

APPLIED SCIENCES

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(Working Papers)

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Study of Jet Propulsion system: Ramjet Engine

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Abstract: *In this paper we are studying the different forms of Euler equations such as conservative form, primitive form and other forms; as well as representing them in vector-matrix notation. Further we have studied about the propulsion, the reason for jet propulsion, types of jet propulsion and learn in detail about Ramjet engine. We study the principle of operation of Ramjet engine, its pros and cons, its applications and consider some of the examples based on its propulsion in order to find the efficiency of ideal cycle, velocity, air flow rate, fuel-air ratio, propulsive efficiency, thrust, density and temperature.*

Keywords: *Euler equation, Vector-matrix notation, Primitive variable, propulsion system, Rankine-Hugoniot, Ramjet engine, Jet propulsion.*

AMS Subject Classification: *76N15.*

1   Introduction

In general, Fluid mechanics of compressible flow is referred as Gas Dynamics. The study of Gas dynamics is linked with flight, high-speed aircraft, satellites, jets and other space exploration vehicles. Therefore, in applications like this, compressible fluid dynamics is used to obtain solutions of a number of design problems. Fluid mechanics analyzes high speed flows of gases and vapours that are inadequate without considering compressibility. Some of the recent advances made in this area are in transonic, supersonic and hypersonic flows, and unsteady flows in rotating and reciprocating machines, given by Yahya [1]. Laney [2] has broadly analyzed the governing equations of gas dynamics, representing Euler equations in vector-matrix notation and mainly concerns about Euler equations.

Zucker [3] introduces the Mach number as a key parameter and found that in case of a perfect gas, it is very simple to express basic equations of compressible fluid. Liepmann and Roshko [4] began his study of the motion of compressible fluids with the case of one-dimensional flow. The aim of this book is to modernize and extend the treatment of compressible fluid aerodynamics. Anderson [5] has given some history and introductory thoughts on compressible fluid flow and also explained the integral and differential conservation equation for inviscid flows.

Hirsch [6] has determined the solution of the system of Euler equations. He examined the conservation formulation of the Euler equation and has given the Rankine-Hugoniot relations. Oksuzoglu [7] has considered the example of Rankine-Hugoniot from the gas dynamics for shock speed and derivation of the same in terms of jump conditions. Spurk [8] has guided with a good number of exercises designed to develop the ability to model and solve practical problems on steady and unsteady compressible flows. John and Keith [9] has showed that significant conclusions can be drawn concerning the basic differences between incompressible and compressible flows. John [10] aims to foster a deeper understanding of compressible flow

and gas dynamics fundamentals.

There are different types of fluid such as compressible flow where density of the fluid varies from point to point ($\rho \neq$ constant), incompressible flow whose density remains constant ($\rho =$ constant). Sonic flow has Mach number as one that is fluid velocity is same as sound speed. Subsonic flow where Mach number as $0 < M < 0.8$ that is Velocity of the fluid is slower than the speed of sound. Supersonic flow has Mach number as $0.8 < M < 1.4$ so Velocity of the fluid is larger than the speed of sound. In Hypersonic flow Mach number starts from 5 and above that means velocity exceeds the speed of sound.

Some Basic definitions

★ **Mach number:**   Mach number is the ratio of velocity of moving aircraft to speed of sound.

$$M = \frac{u}{c} \tag{1}$$

where, u is flight speed and c is speed of sound.

★ **Mass flow rate:**
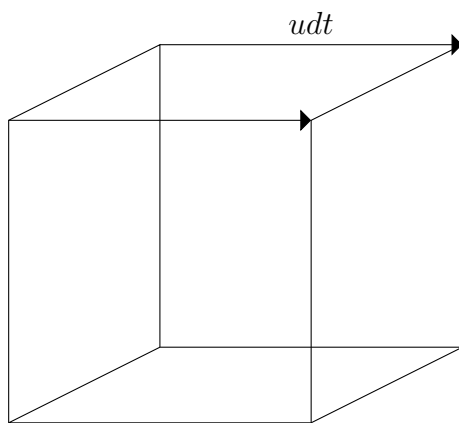
Volume, $dq = Audt$,

Volume flow rate, $\dfrac{dq}{dt} = Au$

$\dot{Q} = Au$

Mass flow rate,

$$dm = \rho dq,$$
$$\dot{m}_a = \rho Au \tag{2}$$

★ **Fuel-air ratio:** Fuel-air ratio is the ratio of fuel to air.

$$\mathbf{FAR} = \frac{\dot{m}_f}{\dot{m}_a} \tag{3}$$

where, $\dot{m}_f$ is mass of fuel and $\dot{m}_a$ is mass of air.

★ **Thrust:** Thrust is a force used to push an object suddenly and strongly. Force is change in momentum with time,

$$F = \frac{mu_2 - mu_1}{t_2 - t_1} \tag{4}$$

Since equation (2) already contains the time dependence, we can express the change in momentum across the propulsion devices as change in mass flow rate times the velocity.

$$F = \dot{m}u_e - \dot{m}_a u \tag{5}$$

where, u_e is exit velocity and u is velocity of aircraft.

★ **Propulsion efficiency:** Propulsion efficiency is the ratio of thrust power to the rate of production of propellent kinetic energy.

$$\eta_p = \frac{Fu}{\dot{m}\left[\dfrac{u_e^2}{2} - \dfrac{u^2}{2}\right]} \tag{6}$$

Substituting for F from equation (5) we get,

$$\eta_p = \frac{2(u_e - u)u}{u_e^2 - u^2},$$

$$\eta_p = \frac{2u}{u_e + u} \tag{7}$$

★ **Stagnation temperature:** Stagnation temperature is the temperature at a stagnation point in a fluid flow. Then for Adiabatic process the energy equation by ignoring the heat, work and gravitational potential energy terms we have,

$$h_0 = h + \frac{u_e^2}{2} \tag{8}$$

where, h_0 is stagnation enthalpy at stagnation point, h is static enthalpy at point of interest along stagnation streamlines with $h = c_p T$ and u_e is exit velocity.

$$\text{Hence, } T_0 = T + \frac{u_e^2}{2c_p} \tag{9}$$

★ **Efficiency of ideal cycle:** Ideal efficiency of Ramjet engine is given by,

$$\eta_i = 1 - \frac{1}{t} \tag{10}$$

Here, t is the temperature ratio given by equation (9) and substituting $c_p = \dfrac{\gamma R}{\gamma - 1}$ and $c = \sqrt{\gamma R T}$ we get,

$$\eta_i = \frac{1}{1 + \dfrac{2}{\gamma - 1}\dfrac{1}{M_i^2}} \tag{11}$$

Basic Equations of Gas Dynamics:

- Bernoulli's equation for compressible flow :
$$\frac{u_1^2}{2} + \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} = \frac{u_2^2}{2} + \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1}\left(\frac{p_2}{p_1}\right)^{\frac{(\gamma - 1)}{\gamma}}.$$
- Equation of state of ideal gas : $\rho_2 = \dfrac{p_2}{RT_2}$
- The energy equation :
$$\frac{u_2^2}{2} + h_2 + q = \frac{u_3^2}{2} + h_3 \quad \text{with } h = c_p T \tag{[4]}$$
- Isentropic relation :
$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma - 1}{\gamma}} \tag{[4]}$$

2   Basic Equations in Vector notation

Mass, momentum and energy are called conserved quantities. Define a vector of conserved quantities as,

$$u = \begin{bmatrix} \rho \\ \rho u \\ \rho e_T \end{bmatrix} \tag{12}$$

where, $u_1 = \rho$ (mass per unit volume), $u_2 = \rho u$ (momentum per unit volume), $u_3 = \rho e_T$ (energy per unit volume).

Let f be the flux with components f_1, f_2, f_3 where, $f_1 = \rho u$ (mass flux), $f_2 = \rho u^2 + p$(momentum flux with pressure), $f_3 = (\rho e_T + p)u$(total energy flux with pressure). Then the flux vector is given by,

$$f = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (\rho e_T + p)u \end{bmatrix} \tag{13}$$

(or)

$$f = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho h_T u \end{bmatrix} \tag{14}$$

where, $h_T = e_T + \frac{p}{\rho}$ (Specific total enthalpy)

Separating the flux and pressure contributions from (13) we have,

$$f = \begin{bmatrix} \rho u \\ \rho u^2 \\ \rho u e_T \end{bmatrix} + \begin{bmatrix} 0 \\ p \\ pu \end{bmatrix} \tag{15}$$

Using the vector notation, the Conservation form can be written as,

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0. \tag{16}$$

The flux vector f can be written as a function of the conserved quantities u.

$u_1 = \rho, \ u_2 = \rho u, \ u_3 = \rho e_T, \ f_1 = \rho u, \ f_2 = \rho u^2 + p, \ f_3 = (\rho e_T + p)u$

Consider,

$$f_2 = \rho u^2 + p,$$
$$= \rho u^2 + (\gamma - 1)(\rho e_T - \frac{1}{2}\rho u^2),$$
$$= \frac{1}{2}\frac{\rho^2 u^2}{\rho}(3 - \gamma) + (\gamma - 1)\rho e_T,$$
$$f_2 = \frac{1}{2}\frac{u_2^2}{u_1}(3 - \gamma) + (\gamma - 1)u_3. \tag{17}$$

Similarly we can find,

$$f_1 = u_2. \tag{18}$$

$$f_3 = \frac{\gamma u_2 u_3}{u_1} - \frac{(\gamma - 1)}{2}\frac{u_2^3}{u_1^2}. \tag{19}$$

Applying chain rule for $\frac{\partial f}{\partial x}$ and substituting in (16) we get,

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0. \tag{20}$$

where, A is $\frac{\partial f}{\partial u}$ and is called the Jacobian matrix of f.

Differentiating f_1, f_2, f_3 with respect to u_1, u_2, u_3 and substituting in A we get,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{(\gamma-3)}{2}u^2 & (3-\gamma)u & \gamma-1 \\ -\gamma u e_T + (\gamma-1)u^3 & \gamma e_T - \frac{3}{2}(\gamma-1)u^2 & \gamma u \end{bmatrix}. \tag{21}$$

We know that,

$$h_T = \frac{1}{2}u^2 + \frac{1}{\gamma-1}a^2. \tag{22}$$

$$\gamma e_T = \frac{\gamma}{2}u^2 + \frac{1}{\gamma-1}a^2. \tag{23}$$

Solving (22) and (23) and substituting for $\frac{\partial f_3}{\partial u_1}$ and $\frac{\partial f_3}{\partial u_2}$ in (21) we get,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{(\gamma-3)}{2}u^2 & (3-\gamma)u & \gamma-1 \\ -uh_T + \frac{1}{2}(\gamma-1)u^3 & h_T - (\gamma-1)u^2 & \gamma u \end{bmatrix}. \tag{24}$$

3 The primitive variable form of the Euler equation

Primitive variable form is another differential form of the Euler equations. Consider the substantial or material derivative,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}. \tag{25}$$

Using (25) we can find the primitive variable form of the following governing equations,

1. **Conservation of Mass**
   The Continuity equation is given by,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0,$$

$$\frac{\partial \rho}{\partial t} + (\nabla \cdot \vec{u})\rho + \vec{u} \cdot \nabla\rho = 0,$$

$$\frac{D\rho}{Dt} + \rho\frac{\partial u}{\partial x} = 0. \tag{26}$$

2. **Conservation of momentum**

$$\frac{Du}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0. \tag{27}$$

3. **Conservation of Energy**

$$\frac{Dp}{Dt} + \rho a^2\frac{\partial u}{\partial x} = 0. \tag{28}$$

4. **The second law of Thermodynamics**

$$\frac{Ds}{Dt} \geq 0. \tag{29}$$

Equations (26), (27), (28) and (29) are the primitive variable form of the Euler equations.
The primitive variable form of Euler equations (26), (27), (28) can also be written in vector
matrix form as,

$$\frac{\partial w}{\partial t} + C\frac{\partial w}{\partial x} = 0. \tag{30}$$

where, $\quad w = \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}$, $C = \begin{bmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & \rho a^2 & u \end{bmatrix}$ and we have, $u = \begin{bmatrix} \rho \\ \rho u \\ \rho e_T \end{bmatrix}$

$$\text{we notice that, } Q = \frac{du}{dw} = \begin{bmatrix} 1 & 0 & 0 \\ u & \rho & 0 \\ \frac{1}{2}u^2 & \rho u & \frac{1}{\gamma - 1} \end{bmatrix}. \tag{31}$$

where, $du = \dfrac{\partial u}{\partial x}$ and $dw = \dfrac{\partial w}{\partial x}$.
Hence we have $du = Qdw$ and similarly $dw = Q^{-1}du$.

$$\text{where, } Q^{-1} = \frac{dw}{du} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{\rho}u & \frac{1}{\rho} & 0 \\ \frac{1}{2}(\gamma - 1)u^2 & -(\gamma - 1)u & \gamma - 1 \end{bmatrix}. \tag{32}$$

Substituting ∂u in (20) we get,

$$Q\frac{\partial w}{\partial t} + AQ\frac{\partial w}{\partial x} = 0.$$

$$\frac{\partial w}{\partial t} + Q^{-1}AQ\frac{\partial w}{\partial x} = 0. \tag{33}$$

Comparing with equation (30) we get,

$$C = Q^{-1}AQ. \tag{34}$$

where, A and C are similar matrices.

4 Other form of the Euler equation

Euler equations are mainly written using Conservative form and the Primitive variable form.
Other forms of Euler equations are formed by multiplying the Euler equation by an invertible 3
$\times$ 3 matrix. Multiplying both sides of the equation (20) by invertible 3×3 matrix Q^{-1} we get,

$$Q^{-1}\frac{\partial u}{\partial t} + Q^{-1}A\frac{\partial u}{\partial x} = 0. \tag{35}$$

In equation (35) we notice that there is no change in dependent variables. But there is change
in equation. Multiplying both sides (30) by Q^{-1},

$$Q^{-1}\frac{\partial w}{\partial t} + Q^{-1}C\frac{\partial u}{\partial x} = 0. \tag{36}$$

In equation (36) there is no change in dependent variables. But there is a change in equation.
Rewrite equation (35) as,

$$Q^{-1}\frac{\partial u}{\partial t} + Q^{-1}AQQ^{-1}\frac{\partial u}{\partial x} = 0. \tag{37}$$

Consider linear change of dependent variables u to v,

$$dv = Q^{-1}du, \tag{38}$$

$$\text{then, } \frac{\partial v}{\partial x} = Q^{-1}\frac{\partial u}{\partial x} \text{ and } \frac{\partial v}{\partial t} = Q^{-1}\frac{\partial u}{\partial t}. \tag{39}$$

Substituting (39) in equation (37) we get,

$$\frac{\partial v}{\partial x} + Q^{-1}AQ\frac{\partial v}{\partial t} = 0. \tag{40}$$

5   Ramjet engine

Figure 1: Rene Leduc (1898-1968)

Rene Leduc was a French engineer, self educated and multitalented person. He was publicly praised for his major achievements on Ramjets. In one of the Aerostories by Philippe Ricco [11] he has mentioned that, in 1913, Rene Lorin first published the principle in the technical review "Aerophile" and invented the Ramjet in France. Unfortunately, he could not build his invention. In 1933, Rene Leduc became the first aeronautical designer to successfully develop the Ramjet engine. He discovered Lorin's publications and paid great respect and honor to Lorin's work. After months of efforts, he was able to clearly show the existence of the practical application of this theory in the year 1936.

John and Keith [9] explain compressible flow, isentropic flow through nozzle and explains the variation in stagnation properties that defines the reference state of compressible flow and stagnation pressure in steady isentropic flow. Yahya [1] has given the basic definitions like mach number, stagnation pressure ratio, isentropic relation and involves problems of compressible flow. Problems have been specially chosen in the areas of aerospace, chemical, gas and mechanical engineering. Roy and Pradeep [12] have explained in their lectures about the basic aspects of jet aircraft propulsion. The focus is primarily on the aircraft propulsion mode that is the air breathing mode of propulsion.

Hill and Peterson [13] have proceeded under the principle that a few fundamental physical principles can with suitable application gives an understanding of all aspects of aircraft and spacecraft propulsion. Furthermore, explained about the jet propulsion and air breathing engines. Krueger [14] detailed about propulsive efficiency as a key indicator of propulsive performance, but it can be difficult to measure when the propulsion system is integrated into the vehicle body because the average rate of useful work done propelling the vehicle and/or the average mechanical power expended propelling the vehicle is not known directly. Whittle [15] has briefly told about the early history of the development of the jet propulsion gas turbine in Great Britain, concentrating chiefly on the story up to the first flight tests in May 1941.

6   Jet propulsion system

Propulsion: Propulsion is derived from latin words: Pro means backward or forward; Pellere means to drive. Hence propulsion means to drive an object forward.

Propulsion system: Propulsion system is a system that converts mechanical power into propulsive force.

Propeller $(M\Delta v)$ **:** Propeller is a device that is used to push the aircraft forward or backward. It has low subsonic flight speed. To overcome this obstacle Jet Propulsion was introduced.

Jet propulsion system is a propulsion of jet aircraft which uses atmospheric air. Jet Propulsion is a practical application of Newton's third law of motion. "For every force acting on a body there is an equal and opposite reaction". The Jet engine accelerates mass in one direction and experiences thrust in opposite direction. It uses air to burn fuel. In contrast to propeller, Jet propulsion system imparts a large velocity change to small mass air so that large amount of thrust can be produced to improve flight speed to high subsonic flight speeds.

Types of jet propulsion system

The jet propulsion system are classified based on the method of operation. One is atmospheric jet engine and the other is rockets. They are: Ramjet, Scramjet, Turbojet, Turbofan, Turboprop, Pulsejet engine.

7   Ramjet engine
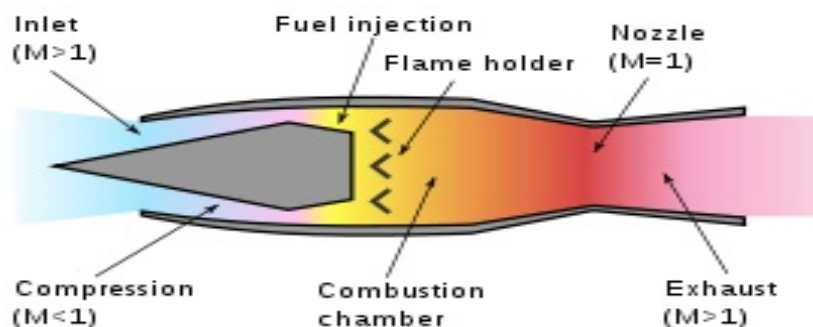
Figure 2: Design of a Ramjet engine

Ramjet engine is most basic type of jet engine available. It is an air breathing engine hence it uses oxygen as oxidiser and hydrogen as propellent. It compress air supersonic speed to subsonic speed. Though this engine has no moving parts inside it, there are only three parts:

CONVERGING INLET: Air is compressed to a very high ratio and sent inside the engine.

COMBUSTION CHAMBER: Fuel is injected and flame holder burn the fuel-air mix. Since it have high enthalpy they flow rapidly towards nozzle.

A NOZZLE: Air squeezes out through nozzle and there is increase in velocity due to thrust.

8   Principle of working of the Ramjet
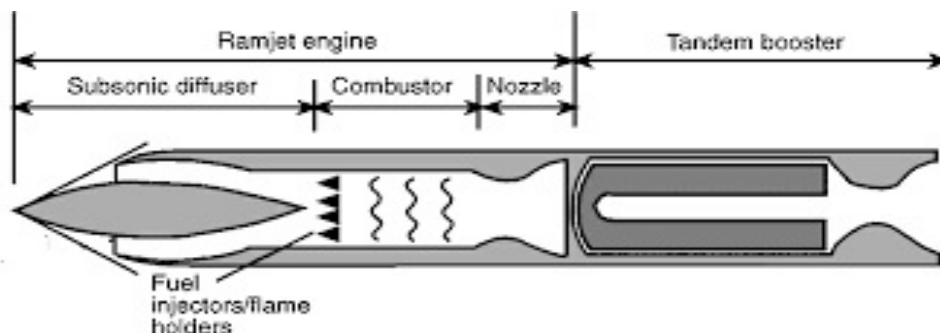
Figure 2: Working of Ramjet engine

Atmospheric air enters the supersonic diffuser which reduces the velocity and increases the static pressure. Further it passes through the subsonic diffuser where the velocity reduces to subsonic value and the pressure of air increases to ignition pressure. This pressure then flows to the combustion chamber where air-fuel mixture is burnt. Later the highly heated products are allowed to expand in the exhaust nozzle where the energy of gas is converted into kinetic energy with very high velocity. This type of transformation in energy is called the **_Ram effect_** and the rise in pressure is called **_Ram pressure_**. Hence thrust is produced in opposite direction which helps in propulsion of aircraft.

9   The study of efficiency of Ramjet engine

Considering a Ramjet engine propulsion in order to calculate the efficiency of ideal cycle, flight speed, air flow rate, fuel-air ratio, propulsive efficiency and thrust.

Case 1: Propulsion taking place at mach number 1.4 with an initial temperature and pressure $249.5K$ and $0.472 \times 10^5 N/m^2$. The radius of the inlet diffuser at entry is $20cm$. The stagnation temperature at the nozzle entry is $1500k$ with exit temperature $1129.5K$ and fuel mass $1.015Kg/s$. The property of combustion gases are same as that of air.($\gamma = 1.4, R = 287J/kgK$)

(i) **The efficiency of ideal cycle**

$$\eta_i = \frac{1}{1 + \frac{2}{\gamma-1} \times \frac{1}{M_1^2}}$$

$\eta_i = 0.281 = 28.1\%$

(ii) **Flight speed**

Mach number, $M_1 = \dfrac{u}{c}$,   where $c = \sqrt{\gamma RT}$

$u = 443.1m/s$

(iii) **Air flow rate**

$\dot{m}_a = \rho_1 A_1 u$,   where $A_1 = \dfrac{\pi}{4}d^2$ and $\rho_1 = \dfrac{p_1}{RT_1}$

$\dot{m}_a = 36.56Kg/s$

(iv) **Fuel-air ratio**

$\text{FAR} = \dfrac{\dot{m}_f}{\dot{m}_a}$

FAR $= 0.027$

(v) **Propulsion efficiency**

$$\eta_p = \frac{2u}{u_e + u}$$

we have, $T_0 = T_e + \frac{u_e}{2c_p}$ and $c_p = \frac{\gamma R}{\gamma - 1}$

$\eta_p = 0.678 = 67.8\%$

(vi) **Thrust**

$$F = \dot{m}u_e - \dot{m}_a u, \quad \text{where } \dot{m} = \dot{m}_a + \dot{m}_f$$

$$F = 16.22 \times 10^3 N$$

Case 2 : Taking the following experimental values into consideration

$M_1 = 1.5, \gamma = 1.4, R = 287 J/KgK, d = 50cm, T_1 = 245.90K, p_1 = 0.440bar, \dot{m}_f = 1.841Kg/s, T_0 = 1600K, T_e = 1163.027K$

(i) **The efficiency of ideal cycle**

$\eta_i = 0.3103$

(ii) **Flight speed**

$u = 471.48m/s$

(iii) **Air flow rate**

$\dot{m}_a = 57.66Kg/s$

(iv) **Fuel-air ratio**

FAR $= 0.0319$

(v) **Propulsion efficiency**

$\eta_p = 0.6694 = 67\%$

(vi) **Thrust**

$F = 28.572 \times 10^3 N$

Case 3 : Considering a Ramjet engine propulsion with isentropic flow, isobaric heat addition in order to calculate the exit velocity,upstream and downstream velocity of the combustion chamber, temperature, density using the below experimental values   [8]

$u_1 = 300m/s, p_2 = p_3 = 1.25bar = 1.25 \times 10^5 Pa, \gamma = 1.4, p_0 = p_1 = p_4 = 0.8bar = 0.8 \times 10^5 Pa, q_{23} = 300kJ/Kg, R = 287 J/KgK, T_1 = 273K, A_1 = 1m^2$

(i) **Upstream velocity**

We have Bernoulli's equation for compressible flow as

$$\frac{u_1^2}{2} + \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} = \frac{u_2^2}{2} + \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1}\left(\frac{p_2}{p_1}\right)^{\frac{(\gamma-1)}{\gamma}}$$

$$\frac{p_1}{\rho_1} = RT_1$$

$$u_2 = \sqrt{u_1^2 + \frac{2\gamma}{\gamma - 1}RT_1\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$

$u_2 = 124.14m/s$

(ii) **Temperature T_2 and density ρ_2**

We have isentropic relation as

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}$$

$$T_2 = T_1\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}$$

$T_2 = 310.13K$

We have equation of state of ideal gas as,

$$\rho_2 = \frac{p_2}{RT_2}$$

$$\rho_2 = 1.4044 Kg/m^3$$

(iii) **Temperature T_3 and density ρ_3**

Since heat addition is isobaric and it is inviscid fluid there is no pressure changes and no particle acceleration.

Hence $u_2 = u_3$

The energy equation is given by,

$$\frac{u_2^2}{2} + h_2 + q = \frac{u_3^2}{2} + h_3$$

with $h = c_p T$

$$q_{23} = c_p(T_3 - T_2)$$

using specific heat at constant pressure,

$$c_p = \frac{\gamma R}{\gamma - 1}$$

hence, $q_{23} = \dfrac{\gamma R}{\gamma - 1}(T_3 - T_2)$

$$T_3 = q_{23}\frac{\gamma - 1}{\gamma R} + T_2$$

$$T_3 = 608.79 K$$

and $\rho_3 = 0.7154 Kg/m^3$

(iv) **Exit velocity u_4 and density ρ_4**

Using the Bernoulli's equation of compressible flow we get,

$$u_4 = \sqrt{u_3^2 + \frac{2\gamma}{\gamma - 1}RT_3\left[1 - \left(\frac{p_4}{p_3}\right)^{\frac{\gamma - 1}{\gamma}}\right]}$$

$$u_4 = 402.28 m/s$$

Isentropic relation is given by,

$$\frac{\rho_4}{\rho_3} = \left(\frac{p_4}{p_3}\right)^{\frac{1}{\gamma}}$$

$$\rho_4 = 0.5201 Kg/m^3$$

Disadvantages of Ramjet engine

- Take-off thrust is zero. Hence to start Ramjet an external launching device is needed.
- The combustion chamber requires flame holder to stabilize the combustion because of the high air speed.
- At lower speeds, the propulsive efficiency and thrust are lower.
- Sudden decrease of speed is difficult.

10 Applications of Ramjet engine

- Used in high speed air crafts and missiles.
- Subsonic Ramjets are used in target weapons.

Figure 3: Test firing of the Akash missile from Integrated Test Range (ITR), Chandipur, Odisha

The Defence Research and Development Organization (DRDO) developed a medium-range mobile surface-to-air missile defense system called "Akash". It was produced by Bharat Electronics (BEL) for other radars and Bharat Dynamics Limited (BDL) for Missile Systems, control centers in India. This missile system targets aircraft and has the capability to neutralize air targets like fighter jets, air-to-surface missiles and cruise missiles. It is operated by Indian Army and Indian Air Force. As mentioned in DRDO.gov.in this missile uses a solid propellant which has high-energy and ram-rocket propulsion for the sustainer phase. It has better performance with minimum mass and propulsion system provides high energy. As a result of the successful test of the Akash missile system, "India has achieved the capability of making any type of Surface to Air Missile", the press release reads.

11 Results and Discussions

We have studied different forms of Euler equation and representing them in vector-matrix notation. We have learnt about the jet propulsion system mainly the Ramjet engine and also listed its principle of working. It makes use of high pressure in front of the engine which forces the atmospheric air to pass through the stovepipe, where the air-fuel mixture is burnt. It is then passed through a nozzle to accelerate which gives a forward thrust to the engine. We made use of Bernoulli's equation, energy equation and isentropic relation of compressible flow in order to study the efficiency of propulsion of Ramjet in subsonic flow. We verified the working of Ramjet and its efficiency by using available experimental data.

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Study of Waves and Rocket Propulsion

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Abstract: *We have studied about waves, its types, scalar and vector models of wave equation, the characteristic form of the Euler equation and its applications. Furthermore, we gain knowledge about rocket propulsion, its working, propellants, applications and the efficiency of the propulsion system by considering an example.*

Keywords: *Scalar and Vector model, Euler equations, Simple and Expansion waves, Rocket Propulsion theory.*

AMS Subject Classification: *76N15.*

1 Introduction

In mathematics and other related fields, waves is considered to be a disturbance of a field in which a physical attribute oscillates repeatedly at each point. Tipler [1] says wave motion can be presented as the transport of energy and momentum from one point in space to a different without the transport of matter. There is a theorem [2] which states that, "Simple wave is a flow in a region adjacent to a region of constant state." Compressive waves until shock wave forms are also simple waves. The speed at which small amplitude disturbance are propagated to other parts of the fluid is called the 'acoustic speed or speed of sound'. Rozdestvenskii and Janenko [3] have developed the general theory of quasi-linear systems of equations with application to gas dynamics.

2 Waves for a scalar model problem

Consider a first order partial differential equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \tag{1}$$

where $u = u(x,t)$.

- If $a = $ constant, then equation (1) becomes linear
- If $a = a(u,x,t)$, then equation (1) becomes quasi-linear.

Linear or quasi-linear, equation (1) has wave solution.

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = (1,a)\left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}\right) = (1,a).\nabla u.$$

There is no change in the solution u in the direction of $(1,a)$ in $x - t$ plane.
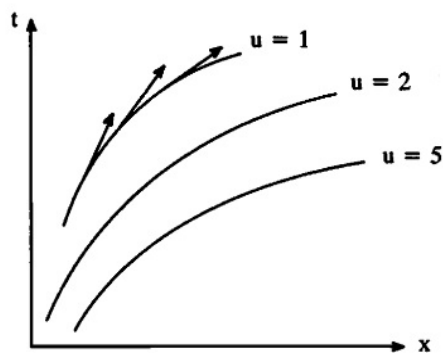
Figure 1: A typical wave diagram for a scalar model problem

Consider the curve which is tangent to $(1, a)$ in the (x, t) plane. Let $x = x(t)$ where the slope of the vector $(1, a)$ is a and the slope of the curve $x = x(t)$ is $\dfrac{dx}{dt}$. Then, we get

$$u = \text{constant for } \frac{dx}{dt} = a \tag{2}$$

where the curve $\dfrac{dx}{dt} = a$ is called as wavefront or characteristics curve or simply characteristics, u is the signal or wave information and a is the wave speed.

3   Waves for a vector model problem

Consider a system of first order partial differential equation

$$\frac{\partial u}{\partial t} + A \frac{\partial u}{\partial x} = 0 \tag{3}$$

where $u = u(x, t)$ and A is any square matrix. The system of equation (3) is hyperbolic if and only if A is diagonalizable i.e. $Q^{-1}AQ = \Lambda$ where A is diagonal matrix with λ_i's, λ_i are eigen values of A, Q is a matrix containing r_i's in its column, r_i is the right eigen vector of A, Q^{-1} is a matrix containing l_i's in its column, l_i is the left eigen vector of A.

The right characteristic vector is defined as $Ar_i = \lambda_i r_i$ and the left characteristic vector are defined as $l_i^T A = \lambda_i l_i^T$. By left multiplying equation (3) with Q^{-1} we obtain

$$Q^{-1}\frac{\partial u}{\partial t} + Q^{-1}A\frac{\partial u}{\partial x} = 0 \tag{4}$$

Equation (4) is called a characteristic form of equation (3).

The characteristic variable v is defined as

$$dv = Q^{-1}du \tag{5}$$

Then the characteristic form becomes,

$$\frac{\partial v}{\partial t} + \Lambda \frac{\partial v}{\partial x} = 0. \tag{6}$$

Now consider the ith equation in equation (6)

$$\frac{\partial v_i}{\partial t} + \lambda_i \frac{\partial v_i}{\partial x} = 0 \tag{7}$$

where

$$v_i = \text{constant} \ \ \text{for} \ \frac{dx}{dt} = \lambda_i. \tag{8}$$

The curve $dx = \lambda_i dt$ is called wave fronts, the variable v_i is called signals or characteristic variables and λ_i is the wave speed or characteristic speed.
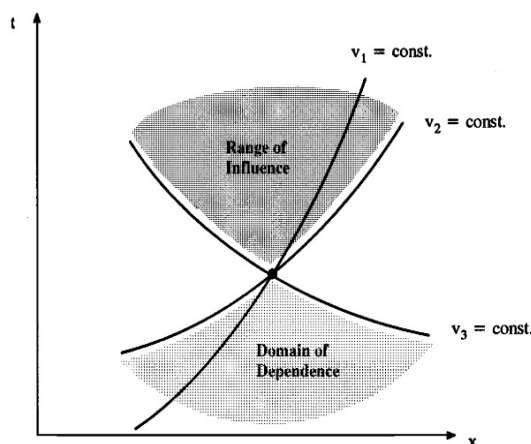
Figure 3: A typical wave diagram for a vector model problem

Here equation (7) says that the ith characteristic variable is constant along the ith characteristic curve. Figure 3 intersections between characteristics of the same family are not any intersection between two characteristics from the same family creates a shock wave. Shock waves are not governed by the ordinary characteristic equations, which originate in differential forms of the governing equations, but instead are governed by jump relations and the theory of weak solutions, which originate in integral forms of the governing equations.

Equation (5) may not always have an analytic solution. Even though dv is always analytically defined, v may not be. Thus equation (8) becomes

$$dv_i = 0 \ \text{for} \ \frac{dx}{dt} = \lambda_i. \tag{9}$$

where $dv_i = 0$ is called as compatibility relations and $\frac{dx}{dt} = \lambda_i$ may not have an analytic solution in some cases. The wave description is of more limited use when either $dv_i = 0$ or $\frac{dx}{dt} = \lambda_i$ lack an analytical solution.

3.1 The characteristic form of the Euler equations

Let us consider the Euler equation in the form $\frac{\partial w}{\partial t} + C \frac{\partial w}{\partial x} = 0$ where

$$w = \begin{bmatrix} \rho \\ u \\ p \end{bmatrix} \ \text{and} \ C = \begin{bmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & \rho a^2 & u \end{bmatrix}$$

Since C is diagonalizable we have $Q_C^{-1}CQ_C = \Lambda$. By right and left characteristic vectors associated with the values of Λ and finding the determinant value of C we get,

$$Q_C = \begin{bmatrix} 1 & \frac{\rho}{2a} & -\frac{\rho}{2a} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{\rho a}{2} & -\frac{\rho a}{2} \end{bmatrix}, \; Q_C^{-1} = \begin{bmatrix} 1 & 0 & -\frac{1}{a} \\ 0 & 1 & \frac{1}{\rho a} \\ 0 & 1 & -\frac{1}{\rho a} \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} u & 0 & 0 \\ 0 & u+a & 0 \\ 0 & 0 & u-a \end{bmatrix}.$$

Then by equation (4), a characteristic form of the Euler equation is given by

$$Q_C^{-1}\frac{\partial w}{\partial t} + Q_C^{-1}C\frac{\partial w}{\partial x} = 0, \tag{10}$$

which can be written as

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} - \frac{1}{a^2}\left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x}\right) = 0,$$

$$\frac{\partial u}{\partial t} + (u+a)\frac{\partial u}{\partial x} + \frac{1}{\rho a}\left(\frac{\partial p}{\partial t} + (u+a)\frac{\partial p}{\partial x}\right) = 0,$$

$$\frac{\partial u}{\partial t} + (u-a)\frac{\partial u}{\partial x} - \frac{1}{\rho a}\left(\frac{\partial p}{\partial t} + (u-a)\frac{\partial p}{\partial x}\right) = 0.$$

Then by equation (6), a characteristic form that involves characteristic rather than primitive variables is given by

$$\frac{\partial v}{\partial t} + \Lambda\frac{\partial v}{\partial x} = 0,$$

$$\frac{\partial v_0}{\partial t} + u\frac{\partial v_0}{\partial x} = 0,$$

$$\frac{\partial v_+}{\partial t} + (u+a)\frac{\partial v_+}{\partial x} = 0, \tag{11}$$

$$\frac{\partial v_-}{\partial t} + (u-a)\frac{\partial v_-}{\partial x} = 0$$

where,

$$dv = Q_C^{-1}dw,$$

$$dv_0 = d\rho - \frac{dp}{a^2},$$

$$dv_+ = du + \frac{dp}{\rho a}, \tag{12}$$

$$dv_- = du - \frac{dp}{\rho a}.$$

Using $p = (\gamma - 2)\left(\rho e_T - \frac{1}{2}\rho u^2\right)$, the Euler equation can be written as

$$dv_0 = d\rho - \frac{dp}{a^2} = 0 \quad \text{for} \quad dx = udt,$$

$$dv_+ = du + \frac{dp}{\rho a} = 0 \quad \text{for} \quad dx = (u+a)dt, \tag{13}$$

$$dv_- = du - \frac{dp}{\rho a} = 0 \quad \text{for} \quad dx = (u-a)dt.$$

Integrating the compatibility relations, the above equations become

$$s = \text{constant} \quad dx = u dt, \tag{14}$$

$$v_+ = u + \int \frac{dp}{\rho a} = \text{constant} \quad dx = (u + a) dt, \tag{15}$$

$$v_- = u - \int \frac{dp}{\rho a} = \text{constant} \quad dx = (u - a) dt. \tag{16}$$

In general, only the first compatibility relation is fully analytically integrable. Also to show that $s = \text{constant}$ for $dx = u dt$ the specific entropy is given by $s = c_v \ln(p) - c_p \ln(\rho) + \text{constant}$, and also we know that

$$\gamma = \frac{c_p}{c_v} \quad \text{and} \quad a^2 = \frac{\gamma p}{\rho}.$$

Thus we get,

$$\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + u \frac{\partial s}{\partial x} = 0 \tag{17}$$

where $\dfrac{D}{Dt}$ is the substantial derivative defined by equation

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \tag{18}$$

Then equations (17) and (14) tell us that the entropy is constant. Comparing equation (14) with equation (18), we observe that these equations are nearly identical: equation $\dfrac{Ds}{Dt} \geq 0$ tells that the substantial derivative of the entropy is greater that or equal to zero where as equation (17) says that the substantial derivative of the entropy is exactly equal to zero.

The characteristic form that the conservation form of the Euler equation can be written as,

$$\frac{\partial u}{\partial t} + A \frac{\partial u}{\partial x} = 0 \tag{19}$$

where

$$u = \begin{bmatrix} \rho & \rho u & \rho e_T \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2}u^2 & (3-\gamma)u & \gamma-1 \\ -\gamma u e_T + (\gamma-1)u^3 & \gamma e_T - \frac{3}{2}(\gamma-1)u^2 & \gamma u \end{bmatrix}.$$

Then $Q_A^{-1} A Q_A = \Lambda$.

By right and left characteristic vectors associated with the values of Λ we get,

$$Q_A = \begin{bmatrix} 1 & \dfrac{\rho}{2a} & -\dfrac{\rho}{2a} \\ u & \dfrac{\rho}{2a}(u+a) & -\dfrac{\rho}{2a}(u-a) \\ \dfrac{u^2}{2} & \dfrac{\rho}{2a}\left(\dfrac{u^2}{2} + \dfrac{a^2}{\gamma-1} + au\right) & -\dfrac{\rho}{2a}\left(\dfrac{u^2}{2} + \dfrac{a^2}{\gamma-1} - au\right) \end{bmatrix} \tag{20}$$

$$\text{and} \quad Q_A^{-1} = \frac{\gamma-1}{\rho a} \begin{bmatrix} \dfrac{\rho}{a}\left(-\dfrac{u^2}{2} + \dfrac{a^2}{\gamma-1}\right) & \dfrac{\rho}{a}u & -\dfrac{\rho}{a} \\ \dfrac{u^2}{2} - \dfrac{a^2}{\gamma-1} & -u + \dfrac{a}{\gamma-1} & 1 \\ -\dfrac{u^2}{2} - \dfrac{a^2}{\gamma-1} & u + \dfrac{a}{\gamma-1} & -1 \end{bmatrix}.$$

Using $h_T = \dfrac{1}{2}u^2 + \dfrac{1}{\gamma - 1}a^2$, equation (20) becomes

$$
Q_A = \begin{bmatrix} 1 & \dfrac{\rho}{2a} & -\dfrac{\rho}{2a} \\[2mm] u & \dfrac{\rho}{2a}(u+a) & -\dfrac{\rho}{2a}(u-a) \\[2mm] \dfrac{u^2}{2} & \dfrac{\rho}{2a}(h_T + au) & -\dfrac{\rho}{2a}(h_T - au) \end{bmatrix},
\tag{21}
$$

such that $dv = Q_A^{-1}du = Q_C^{-1}dw$.

4 Simple waves

Consider a homentropic flow of a perfect gas given by,

$$ p = (\text{constant})\rho^\gamma, $$

$$ a = (\text{constant})\rho^{\dfrac{(\gamma - 1)}{2}} $$

The above expressions can be expressed as $\dfrac{dp}{\rho a}$ entirely in terms of a.

In particular,

$$ \int \frac{dp}{\rho a} = \frac{2a}{\gamma - 1} + \text{constant}, \tag{22} $$

Using equation (22), equations (14) to(16) becomes

$$ s = \text{constant}, $$

$$ v_+ = u + \frac{2a}{\gamma - 1} = \text{constant for } dx = (u + a)dt, $$

$$ v_- = u - \frac{2a}{\gamma - 1} = \text{constant for } dx = (u - a)dt. $$

The characteristic variables $v_\pm = u \pm \dfrac{2a}{\gamma - 1}$ are also known as Riemann invariants, where u and a are constants along the characteristics $dx = (u + a)dt$.

By integrating, we get $x = (u+a)t + \text{constant}$ which shows that the characteristics are straight lines. Thus we have

- Assuming $s = \text{constant}$ and $v_- = u - \dfrac{2a}{(\gamma - 1)} = \text{constant}$ then all flow properties are constant along the characteristic lines

$$ x = (u + a)t + \text{constant}. \tag{23} $$

- Assuming $s = \text{constant}$ and $v_+ = u + \dfrac{2a}{(\gamma - 1)} = \text{constant}$ then all flow properties are constant along the characteristic lines

$$ x = (u - a)t + \text{constant}. \tag{24} $$

- Assuming $v_- = \text{constant}$ and $v_+ = \text{constant}$ then all flow properties are constant along the characteristics lines

$$x = ut + \text{constant.} \tag{25}$$

The above three equations are called a simple waves. The flow regions governed by equation (23) or (24) are called simple acoustic waves and flows governed by equation (25) are called simple entropy waves.

Equation (23) involves only one variable $(u + a)$ and it can be written as

$$\frac{\partial (u + a)}{\partial t} + (u + a)\frac{\partial (u + a)}{\partial x} = 0.$$

Similarly equation (24) can be expressed as

$$\frac{\partial (u - a)}{\partial t} + (u - a)\frac{\partial (u - a)}{\partial x} = 0.$$

Consider the simple entropy waves defined by $v_+ = \text{constant}$ and $v_- = \text{constant}$. If these equations are added and subtracted we obtain $u = \text{constant}$ and $a = \text{constant}$. Hence the velocity and speed of sound are constant throughout a simple entropy waves.

Then the equation (25) implies $\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} = 0$. In terms of density we get, $\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} = 0$.

5   Expansion waves

An expansion wave decreases pressure and density. An expansion wave in any region for a one-dimensional flow of perfect gas with wave speed $\lambda_2 = u + a$ and $\lambda_3 = u - a$ increases monotonically from left to right. More specifically when,

$$u(x, t) + a(x, t) \leq u(y, t) + a(y, t), \quad b_1(t) \leq x \leq y \leq b_2(t),$$
$$u(x, t) - a(x, t) \leq u(y, t) - a(y, t), \quad b_1(t) \leq x \leq y \leq b_2(t).$$
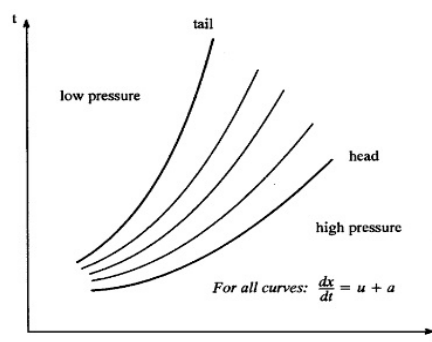
Figure 4: Wave diagram for an expansion in the Euler equation

Figure 4 shows the characteristics in the family creating the expansion. Expansion waves are composed of acoustic and entropy waves which cannot expand and the boundaries $b_1(t)$ and $b_2(t)$ are the characteristics. The boundary on high-pressure is called head of the expansion and the low-pressure is called tail of the expansion.

A simple expansion wave is an expansion wave that is also simple wave which separates regions of steady uniform flow. A centered expansion fan is an expansion wave in which all characteristics originate from a single point in the $x - t$ plane which is an expansion wave that is both simple and centered. The term "fan" is used because it looks like an old-fashioned hand fan in a wave diagram.

5.1   Derivation of the velocity, speed and pressure in the expansion wave

Assume specific entropy $s = $ constant and suppose a simple fan centered at $(x, t) = (0, 0)$ connects the two steady uniform flows u_L and u_R. If $u + \dfrac{2a}{\gamma - 1} = $ constant then,

$$u + \frac{2a}{\gamma - 1} = u_L + \frac{2a_L}{\gamma - 1} = u_R + \frac{2a_R}{\gamma - 1}. \tag{26}$$

Using equation (24), equation (26) will be

$$u + \frac{2}{\gamma - 1}\left(u - \frac{x}{t}\right) = u_L + \frac{2a_L}{\gamma - 1} = u_R + \frac{2a_R}{\gamma - 1}. \tag{27}$$
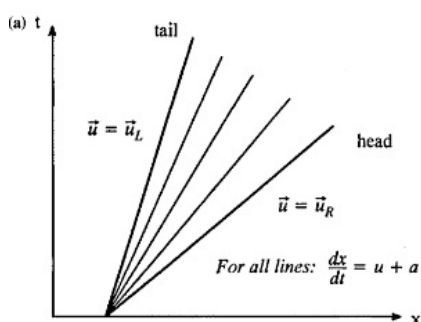
Figure 5: Wave diagram for a simple centered expansion wave in Euler equation

Solving for u and a we get,

$$u(x, t) = \frac{2}{\gamma + 1}\left(\frac{x}{t} + \frac{\gamma - 1}{2}u_L + a_L\right) = \frac{2}{\gamma + 1}\left(\frac{x}{t} + \frac{\gamma - 1}{2}u_R + a_R\right),$$

$$a(x, t) = \frac{2}{\gamma - 1}\left(\frac{x}{t} + \frac{\gamma - 1}{2}u_L + a_L\right) - \frac{x}{t} = \frac{2}{\gamma - 1}\left(\frac{x}{t} + \frac{\gamma - 1}{2}u_R + a_R\right) - \frac{x}{t}$$

which are the solution for the speed of sound in the expansion.
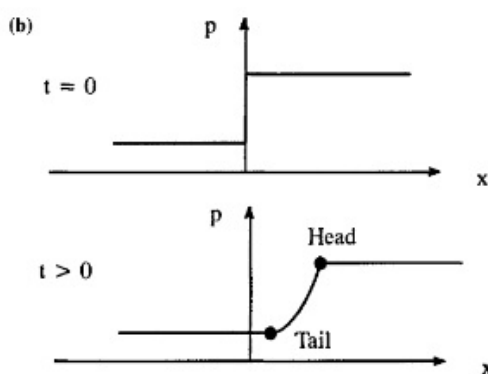
Figure 6: Pressure for a function of x for a simple centered expansion wave in Euler equation

Consider the isentropic relations $p = (const.)\rho^\gamma$

$$p = p_L \left(\frac{a}{a_L}\right)^2 \frac{\gamma}{(\gamma - 1)} = p_R \left(\frac{a}{a_R}\right)^2 \frac{\gamma}{(\gamma - 1)}$$

This is the solution for the pressure in the expansion.

6   Rocket propulsion

Figure 6: Robert H. Goddard (1882-1945)

Robert Hutchings Goddard is an American engineer, professor, physicist and inventor who invented the world's first liquid fueled rocket (1926). He and his team launched 34 rockets between 1929 - 1941, achieving altitudes as high as 2.6km and speeds as fast as 885km/h. Goddard's work in both theorist and engineer anticipated many of the developments that were to make spaceflight possible. A multi-stage rocket (1914) and a liquid-fuel rocket (1914) are his two important inventions which were milestones toward spaceflight. His work in this field was revolutionary but the public support for his research and development work was very less. In 1957, he came to be recognized as one of the founding fathers of modern rocketry [4].

In this section we explore some of the issues surrounding the performance of a whole rocket but have not explored the heart of the rocket, the motors and some of the practical limitations of motor design. A rocket's acceleration depends on three main factors namely
- The greater the exhaust velocity of the gases, the greater the acceleration.
- The faster the rocket burns its fuel, the greater its acceleration.
- The smaller the rocket's mass, the greater the acceleration.

The basic working principle of rocket propulsion is similar to jet propulsion. In rocket propulsion, the altitude of rocket engine is very high and the oxygen required for the combustion is fitted in the tank of the rocket itself. A rocket engine mainly consist of a container for propellants (fuel and oxidizer), combustion chamber (thrust chamber) and a propulsion nozzle. Rocket engines provide essentially the highest specific powers and high specific thrusts of any engine used for spacecraft propulsion.

6.1   Rocket propulsion theory

Some of the main and basic parameters in rocket propulsion are the thrust (F), specific impulse (I_s), specific propellant consumption(Spc) and exhaust gas velocity or effective jet velocity(C_j), pressure ratio across the nozzle and area ratio will effect the above quantities. In energy conservation system, Propulsive efficiency (η_p), thermal efficiency (η_t) and overall

efficiency (η_o) are important.

The exhaust gases in the jet are produced by the propellant. Therefore, the mass rate of flow gases is equal to the ratio of flow of propellants.

$$m_p = m_f + m_{oxi}$$

Thrust (F): The force that propels the rocket at a given velocity is known as the thrust which is produced due to the change in the momentum flux of the outgoing gases as well as difference between the nozzle exit pressure and the anbient pressure. Rocket thrust is employed to overcome the drag and gravitational force besides providing the acceleration.

Consider the development control of the thrust in the rocket where there is no inflow of air into the rocket engine. Let the initial velocity and the final velocity of the gases, before expansion in the nozzle, be $c_1 = 0$ and $c_2 = c_e$ respectively. For a one-dimensional steady flow, at the nozzle exits, the rate of change of momentum flux is given by

$$F_m = \rho_e c_e^2 A_e \tag{28}$$

Continuity equation at the nozzle exits gives

$$m_p = \rho_e A_e c_e \tag{29}$$

Using equation (28), equation (29) becomes

$$F_m = m_p c_e$$

- If the expansion through the nozzle is completely combusted then $P_e = P_a$
- If the expansion through the nozzle is completely combusted then $P_e \neq P_a$

The pressure forces acting on the rocket in the direction of motion is given by $A_e = A_a$, then we have

$$F_{pr} = (P_e - P_a)A_e$$

Net thrust is given by

$$F = m_p c_e + (P_e - P_a)A_e \tag{30}$$

Thrust during the flight through the atmosphere continuously increases with the altitude, an account of decrease in P_a. Under certain altitude the variation is negligible and thrust is almost constant, then equation (30) becomes

$$
\begin{aligned}
F &= m_p c_e \\
&= m_p c_j \quad \left(\text{since } c_e = c_j\right) \\
&= \frac{w_p}{g} c_j \tag{31}
\end{aligned}
$$

Specific impulse (I_s): Specific impulse of a rocket engine is the thrust per unit weight flow rate of the propellant defined by

$$I_s = \frac{F}{w_p}$$

$$= \frac{c_j}{g} \quad \left(\text{Using equation (30)} \right)$$

where c_j depends on the exhaust gas properties.

Specific propellant consumption (Spc): The weight flow rate of the propellant required to produce a thrust of one Newton is known as the specific propellant consumption and is given by $Spc = \dfrac{w_p}{F} = \dfrac{1}{I_s}$.

Effective speed ratio (σ): The ratio of flight speed to jet velocity is known as effective speed ratio and defined as $\sigma = \dfrac{u}{c_j}$.

Propulsive efficiency (η_p): It is defined as the ratio of propulsive power (or thrust power) to the power output of the engine and denoted as $\eta_p = \dfrac{2\sigma}{1+\sigma^2}$, where propulsive power = $F \times u$ = $m_p c_j \times u$, Kinetic energy loss in the exhaust gas = $\dfrac{1}{2}(c_j - u)^2$.

Thus, rate of energy or power loss = $\dfrac{1}{2}m_p(c_p - u)^2$ and power output of the engine =$\dfrac{1}{2}m_p(c_j^2 + u^2)$

Thermal efficiency (η_t): It is defined as the ratio of power output of the engine to the power input to the engine and is given by $\eta_t = \dfrac{c_j^2 + u^2}{2Q_R}$, where power input is given by $m_p Q_R$ and Q_R is the heat of reaction per kg of products of combustion.

Overall efficiency (η_o): It is defined as the ratio of propulsive power to the power input to the engine and defined as $\eta_o = \dfrac{c_j \times u}{Q_R}$.

Example 1: The velocity of a moving rocket is $10,000$ km/hr with $u = 2777.7$ m/sec, with an effective exhaust velocity is $c_j = 1400$ m/sec, the propellant flow rate is $m_p = 5$ kg/sec, the propellant mixture has a heating value of $(C.V) = 6600$ kJ/kg. Find propulsion efficiency, thermal efficiency, overall efficiency and engine output power.

Speed ratio: $\sigma = \dfrac{u}{c_j} = 1.984$

Thrust: $F = m_p \times c_j = 7000$

Propulsive power: $P = F \times u = 19.44 \times 10^6$ W

Propulsion efficiency: $\eta_p = 80.3\%$

Thermal efficiency: $\eta_t = 74.24\%$

Overall efficiency: $\eta_o = 59.6\%$

Engine output power: $= 24.5 \times 10^6$ W

7 Results and Discussions

We have studied about the characteristic form of Euler equation, scalar and vector model problem and its applications. We have learnt about the rocket propulsion and its principle of working. Hot exhaust is produced in the combustion chamber which is then passed through a nozzle which accelerates the flow and produces thrust. Efficiency of the rocket propulsion is obtained by using available experimental data by an example.

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Derivation of Von-Kármán's Integral Equation for Compressible Viscous Fluids and its Application to Boundary Layer Theory

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Abstract: *The boundary layer theory is capable of explaining the difficulties encountered by ideal fluid dynamics and able to predict flow separation. We have studied the steady laminar flow of viscous compressible fluid between two infinite plates separated by a distance using velocity and temperature distribution. The velocity and temperature distribution for Von Kármán-Tsiens's for an insulated plate is plotted in the graphs. We observe that the effect of the Mach number on the compressible viscous flow is to decrease the velocity gradient at the fixed plate and to increase it at the moving plate.*

Keywords: *Compressible Fluid, Boundary Layer Theory, Mach Number, Von Kármán's Integral Equation.*

AMS Subject Classification: *74A06, 76N20, 65R20.*

1   Introduction

The field of fluid mechanics is essential for understanding of many important aspects of applied sciences and engineering streams. It is a subject of widespread interest in almost all fields of engineering as well as biology, meteorology, physical chemistry and geophysics [1]. The development of aeronautical, chemical and mechanical engineering during the past few decades have given added stimuli to the study of fluid mechanics so that it now ranks as one of the most important basic subjects in the study of Engineering Sciences [2].

Fluids play an important role in our day today life in its most fundamental form. At the microscopic level the description of the motion of a fluid involves a study of all the discrete molecules which make up the fluid. In order to get a computational and fundamental understanding of the fluid dynamic problems [3], we must consider the conversational laws in the sense of continuum aspect theory. From the practical application point of view, Computational Problems of fluid dynamics leading to the differential equations and its solutions using boundary layer theory by an approximate method is the **Kármán Pohlhausen Method** based on the **momentum integral equation** which is explained in the following sections. Stewartson has completed the an exhaustive derivation of the complete boundary layer equations governing general compressible flows [4]. In this study Prandtl's boundary layer equations for two dimensional flow of incompressible fluid over a semi-infinite flat plate is encouraged.

2   Prandtl's boundary layer theory

Prandtl introduced boundary layer theory in 1904 to understand the flow behavior of a viscous fluid near a solid boundary. Prandtl gave the concept of a boundary layer in large Reynolds

number flows and derived the boundary layer equations by simplifying the Navier Stokes equations to yield approximate solutions.

Consider laminar two-dimensional flow of fluid of small viscosity over a fixed semi-infinite plate. It is observed that unlike an ideal fluid flow, The fluid does not slide over the plate, but "sticks" to it. Since the plate is at rest, The fluid in contact with it will also be at rest. As we move outwards along the normal, Then velocity of the fluid will gradually increase and at a distance far from the plate the full stream velocity U is attained. Strictly speaking this is approached asymptotically. However, It will be assumed that the transition from zero velocity at the plate to the full magnitude U takes place within a thin layer of fluid in contact with the plate this is known as the Prandtl boundary layer [5]. Prandtl suggested that the entire field of flow can be divided for the sake of mathematical analysis, into the following regions.

- A very thin layer (boundary layer) in the vicinity of gradient normal to the wall, i.e. $\left(\dfrac{\partial u}{\partial y}\right)$ is very large. Accordingly the viscous stress $\mu\left(\dfrac{\partial u}{\partial y}\right)$ becomes an important even when μ small thus the viscous and inertial force are of the same order within the boundary layer.

- In the remaining region (i.e outside the boundary layer) $\dfrac{\partial u}{\partial y}$ is very small and so the viscous forces may be ignored completely outside the boundary layer, The flow can be regarded non-viscous and hence the theory of non-viscous fluids offers a very good approximation in this region.
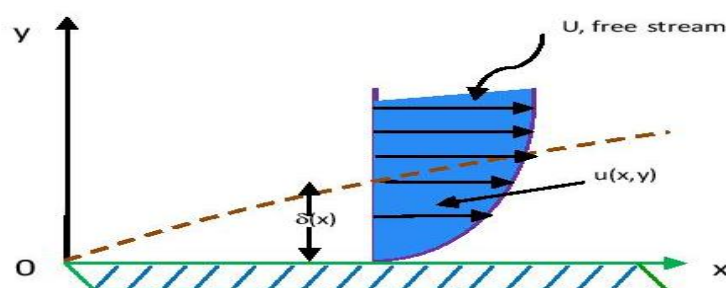
Figure 1

3   Importance of Prandtl's boundary layer theory in fluid dynamics

The boundary layer in Figure 1 is thin,it plays a vital role in fluid dynamics. It has become a very powerful method of analyzing the complex behavior of real fluids. The concept of a boundary layer can be utilized to simplify the Navier-Stokes equations to such an extent that it becomes possible to tackle many practical problems of great importance. The drag on ships and missiles, the efficiency of compressors and turbines in jet engines, The effectiveness of air intakes and turbojets and so on depend on the concept of the boundary layer and its effects on the main flow.

While solving any equation and depending on the arguments in physical terms, The boundary layer theory is capable of explaining the difficulties encountered by ideal fluid dynamics. the boundary layer theory is able to predict flow separation. It can be explained the existence

of a wake. The pressure distribution produces a net force in the direction in which the stream flows, here exist a viscous stress on the boundary region and it acts in the direction of flow.

4   Von Karman's Integral equations to boundary layer

Prandtl's boundary layer equations for two dimensional flow of incompressible fluid over a semi-infinite flat plate are given by

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + v\frac{\partial^2 u}{\partial y^2}, \tag{1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2}$$

with boundary conditions

$$u = v = 0 \text{ when } y = 0 \text{ and } u = U(x, t) \text{ when } y = \infty, \tag{3}$$

we have,

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = -\frac{1}{\rho}\frac{dp}{dx}. \tag{4}$$

Let δ be the boundary layer thickness, Then u, varies from 0 to u as y varies from 0 to δ.

Let τ_δ = shearing stress (at $y = \delta$) = $\left(\mu\dfrac{\partial u}{\partial y}\right)_{y=\delta}$.

Since the fluid is regarded as non-viscous outside the boundary layer, $\tau\delta = 0$ so that $\dfrac{\partial u}{\partial y} = 0$ at $y = \delta$ hence the boundary conditions may be written as

$$u = v = 0, \text{ when } y = 0 \text{ and } u = U \text{ and } \frac{\partial u}{\partial y} = 0, \text{ when } y = \delta. \tag{5}$$

Now, $u\dfrac{\partial u}{\partial x} + u\dfrac{\partial u}{\partial y} = u\dfrac{\partial u}{\partial x} + \dfrac{\partial(uv)}{\partial y} - u\dfrac{\partial v}{\partial y} = u\dfrac{\partial u}{\partial x} + \dfrac{\partial(uv)}{\partial y} - u\left(-\dfrac{\partial u}{\partial x}\right)$ using equation (2).

$$\text{Thus, } u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = 2u\frac{\partial u}{\partial y} + \frac{\partial(uv)}{\partial y} \text{ or } u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial u^2}{\partial x} + \frac{\partial(uv)}{\partial y}. \tag{6}$$

Using equation (6), equation (1) reduces to

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial(uv)}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + v\frac{\partial^2 u}{\partial y^2}. \tag{7}$$

Integrating equation (7) with respect to y from $y = 0$ to $y = \delta$, we get

$$\int\limits_0^\delta \frac{\partial u}{\partial t}dy + \int\limits_0^\delta \frac{\partial u^2}{\partial x}dy = |uv|_0^\delta = -\frac{1}{\rho}\frac{dp}{dx}\int\limits_0^\delta dy + v\left[\frac{\partial u}{\partial y}\right]_0^\delta,$$

where we have used the fact that $\dfrac{dp}{dx}$ is constant across any section. Re-writing and using the boundary conditions (5), the above equation reduces to

$$\frac{\partial}{\partial t}\int\limits_0^\delta udy + \frac{\partial}{\partial x}\int\limits_0^\delta u^2dy + Uv_\delta = -\frac{1}{\rho}\frac{dp}{dx} - \frac{\mu}{\rho}\left(\frac{\partial u}{\partial y}\right)_0 \tag{8}$$

where the suffix δ and 0 signify that the indicated quantities are to be evaluated at $y = \delta$ and $y = 0$ respectively.

Since,

$$\int_0^\delta \frac{\partial v}{\partial y} dy = [uv]_0^\delta = v_\delta \text{ by using equation (5),}$$

we have

$$Uv_\delta = U \int_0^\delta \frac{\partial v}{\partial y} dy = U \int_0^\delta \frac{\partial u}{\partial x} dy, \text{ (using equation (2))} = U \frac{\partial}{\partial x} \int_0^\delta u dy = U \frac{\partial}{\partial x} \int_0^\delta u dy. \quad (9)$$

$$\text{Also, let } \tau_0 = \mu \left(\frac{\partial u}{\partial y} \right)_{y=0} = \text{ shearing stress at the wall.} \quad (10)$$

Using equations (9) and (10), equation (8) becomes

$$\frac{\partial}{\partial t} \int_0^\delta u dy + \frac{\partial}{\partial x} \int_0^\delta u^2 dy - U \int_0^\delta \frac{\partial u}{\partial x} dy = \frac{\delta}{\rho} \frac{dp}{dx} - \frac{\tau_0}{\rho},$$

$$\frac{\partial}{\partial t} \int_0^\delta u dy + \frac{\partial}{\partial x} \int_0^\delta u^2 dy - U \frac{\partial}{\partial x} \int_0^\delta u dy = -\frac{\delta}{\rho} \frac{dp}{dx} - \frac{\tau_0}{\rho}. \quad (11)$$

This is one of the Von-Kármán's integral equation.

5 Plane Couette flow of a Compressible Viscous fluid

Consider the steady laminar flow of viscous Compressible fluid between two infinite plates separated by a distance h. Let the upper plate move with uniform velocity U and the lower plate be at rest. Let x be the direction of flow, y The direction perpendicular to the flow, And the width of the plates parallel to the Z-direction. Here the word 'infinite' implies that the width of the plates is large compared with h and hence the flow may be treated as two-dimensional, i.e. $\frac{\partial}{\partial z} = 0$ [6].
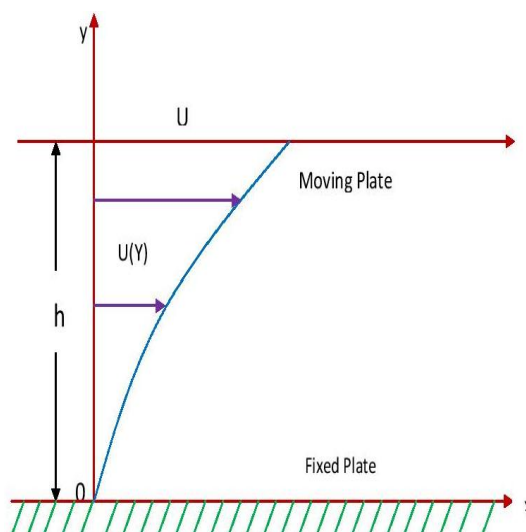

Figure 2: Configuration of Plane Couette flow of a Compressible viscous fluid

Let the plates be long enough in the $x-$direction for the flow to be parallel so that the velocity components V and W are zero everywhere moreover the flow being steady.

The flow variables are independent of time $\left(\dfrac{\partial}{\partial t} = 0 \right)$. Thus, for the flow under consideration, we have

$$\vec{q} = (u, v, w), \quad [u = u(y), v = 0, w = 0], \quad \frac{\partial}{\partial z} = 0, \quad \frac{\partial}{\partial t} = 0, \tag{12}$$

here p is constant.

The basic equations for steady flow of compressible viscous fluid are given by,

Equation of continuity:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho w)}{\partial t} = 0, \tag{13}$$

Navier Stokes equations [3]:

$$\rho \left\{ \frac{\partial u}{\partial t} + u \left(\frac{\partial u}{\partial x} \right) + v \left(\frac{\partial u}{\partial y} \right) + w \left(\frac{\partial u}{\partial z} \right) \right\} = \rho B_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left[\mu \left\{ 2 \frac{\partial u}{\partial x} \right. \right.$$
$$\left. \left. - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right\} \right] + \frac{\partial}{\partial y} \left\{ \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right\} + \frac{\partial}{\partial z} \left\{ \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right\}, \tag{14}$$

$$\rho \left\{ \frac{\partial v}{\partial t} + u \left(\frac{\partial v}{\partial x} \right) + v \left(\frac{\partial v}{\partial y} \right) + w \left(\frac{\partial v}{\partial z} \right) \right\} = \rho B_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial y} \left[\mu \left\{ 2 \frac{\partial v}{\partial y} \right. \right.$$
$$\left. \left. - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right\} \right] + \frac{\partial}{\partial z} \left\{ U \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right\} + \frac{\partial}{\partial x} \left\{ \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right\}, \tag{15}$$

$$\rho \left\{ \frac{\partial w}{\partial t} + u \left(\frac{\partial w}{\partial x} \right) + v \left(\frac{\partial w}{\partial y} \right) + w \left(\frac{\partial w}{\partial z} \right) \right\} = \rho B_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial z} \left[\mu \left\{ 2 \frac{\partial w}{\partial z} \right. \right.$$
$$\left. \left. - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right\} \right] + \frac{\partial}{\partial x} \left\{ \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right\} + \frac{\partial}{\partial y} \left\{ \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right\}, \tag{16}$$

Energy equation:

$$\rho \left\{ \frac{\partial(C_p T)}{\partial t} + u \frac{\partial(C_p T)}{\partial x} + v \frac{\partial(C_p T)}{\partial y} + w \frac{\partial(C_p T)}{\partial z} \right\} =$$
$$\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} + w \frac{\partial p}{\partial z} + \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right)$$
$$+ \mu \left[2 \left\{ \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 \right\} - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)^2 \right.$$
$$\left. + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)^2 \right], \tag{17}$$

Equation of state:

$$P = K \rho T. \tag{18}$$

$$\text{From equation (14)} \implies \frac{d}{dy} \left(\mu \frac{du}{dy} \right) = 0. \tag{19}$$

$$\text{From equation (15)} \implies \frac{d}{dy} \left(k \frac{dT}{dy} \right) + \mu \left(\frac{du}{dy} \right)^2 = 0. \tag{20}$$

The boundary conditions are given by

$$y = 0: \ u = 0, \ T = T_w, \ Q = Q_w \left. \begin{matrix} \\ \\ \end{matrix} \right\}$$
$$y = h: \ u = U, \ T = T_\infty$$

(21)

where Q denotes heat flux, Q_w denotes heat flow to the plate $y = 0, T_w$ and T_∞ respectively denote temperature at moving and stationary plates. Integrating equation (19), we get

$$\mu \left(\frac{du}{dy} \right) = A,$$

(22)

where A is an arbitrary constant. Let shearing stress at the lower plate be given by

$$\left[\mu \left(\frac{du}{dy} \right) \right]_{y=0} = \tau_w, \ \text{say.}$$

(23)

Putting $y = 0$ in equation (22) and using equation (23), we get $A = \tau_w$, then equation (22) becomes

$$\mu \left(\frac{du}{dy} \right) = \tau_w \ \text{so that} \ du = \tau_w \times \left(\frac{dy}{\mu} \right).$$

Integrating, we get

$$\int_{u=0}^{u} du = \int_{y=0}^{y} \tau_w \frac{dy}{\mu} \quad \text{(or)} \quad u = \tau_w \int_{0}^{y} \frac{dy}{\mu}.$$

(24)

Substituting the value of $\dfrac{du}{dy}$ given by equation (23) in equation (20), we get

$$\frac{d}{dy} \left(k \frac{dT}{dy} \right) + \mu \times \left(\frac{\tau_w}{\mu} \right)^2 = 0,$$

$$\frac{d}{dy} \left(k \frac{dT}{dy} \right) = -\frac{\tau_{w^2}}{\mu}.$$

(25)

Integrating, we get

$$k \frac{dT}{dy} = -\tau_{w^2} \int_{0}^{y} \frac{dy}{\mu} + B,$$

where B being an arbitrary constant,

$$k \left(\frac{dT}{dy} \right) = -u\tau_w + B, \ \text{using equation (24),}$$

(26)

$$\left(\frac{kdT}{dy} \right)_{y=0} = -Q_w.$$

(27)

The negative sign is conventional, Q_w is positive when $\left(\dfrac{dT}{dy} \right)_{y=0}$ is negative using the conditions $y = 0, u = 0$.

Equation (27) $\implies u\tau_w + B$ if $B = 0$. Thus, $B = -Q_w$.

Equation (26) $\implies k\dfrac{dT}{dy} + \mu\dfrac{du}{dy} = -Q_w$.

$$k \frac{dT}{dy} = -\frac{1}{2} \mu \frac{du^2}{dy} - Q_w.$$

(28)

We know that the relationship between the co-efficient of Viscosity μ and the co-efficient of thermal conductivity k.

It has been shown that with sufficient accuracy μ can be expressed as a power of T in the form

$$\frac{\mu}{\mu_\infty} = \left(\frac{T}{T_\infty}\right)^m , \text{ where } 0.5 < m < 1, \tag{29}$$

where μ_∞ is the value of μ at the upper moving plate. For air at ordinary temperature we usually take $m = 0.76$. As temperature increases, m decreases towards 0.5.

$$\text{Prandtl number: } P_r = \frac{(C_p\mu)}{k}. \tag{30}$$

Since P_r is very nearly constant for all common gases and C_p is also nearly constant for a fairly wide range of temperatures around ordinary temperature, re-writing equation (30), we get

$$k = \frac{(C_p\mu)}{P_r} \tag{31}$$

Substituting equation (31) in equation (28), we get

$$\frac{C_p\mu}{P_r}\frac{dT}{dy} = -\frac{\mu}{2}\frac{du^2}{dy} - Q_w,$$

(or)

$$\frac{d}{dy}\left(\frac{C_pT}{P_r} + \frac{u^2}{2}\right) = -\frac{Q_w}{\mu}. \tag{32}$$

Integrating, we get

$$\frac{C_pT}{P_r} + \frac{u^2}{2} = -Q_w\int_0^y \frac{dy}{\mu} + \frac{C}{P_r}$$

where C is an arbitrary constant,

$$C_pT + \frac{1}{2}P_ru^2 = -Q_wP_r\int_0^y \frac{dy}{\mu} + C \tag{33}$$

Using the boundary conditions at the fixed plate, namely, when $y = 0$, $u = 0$ and $T = T_w$, equation (33) yields $C = C_pT_w$.

Using (24), equation (33) may be re-written as

$$C_pT + \frac{1}{2}P_ru^2 = -\frac{P_rQ_w}{\tau_w}u + C, \tag{34}$$

which gives relation between u and T.

Determination of velocity and temperature distribution

Using the boundary conditions: When $y = h, u = U, T = T_\infty$, equation (34) gives

$$C_pT_\infty + \frac{1}{2}P_ru^2 = -\frac{P_rQ_w}{\tau_w}u + c, \tag{35}$$

Subtracting equation (34) from equation (33) and simplifying, we obtain

$$C_p(T - T_\infty) = \frac{P_r Q_w}{\tau_w}(U - u) + \frac{1}{2}P_r(U^2 - u^2). \tag{36}$$

Divide equation (36) by $C_p T_\infty$, we get

$$\frac{T}{T_\infty} = 1 + \frac{P_r Q_w U}{C_p T_\infty \tau_w}\left(1 - \frac{u}{U}\right) + \frac{P_r U^2}{2C_p T_\infty}\left(1 - \frac{u^2}{U^2}\right) \tag{37}$$

$$\text{Eckert Number: } E_c = \frac{U^2}{C_p T_\infty} = (\gamma - 1)M_\infty{}^2, \tag{38}$$

where M_∞ denotes the Mach number of the free stream and γ the ratio of specific heats. We have,

$$\frac{U}{C_p T_\infty} = \{(\gamma - 1)M_\infty{}^2\}/U. \tag{39}$$

Substituting equation (39) in equation (37) gives

$$\frac{T}{T_\infty} = 1 + \frac{P_r Q_w}{\tau_w}\frac{(\gamma - 1)}{U}M_\infty{}^2\left(1 - \frac{u}{U}\right) + P_r\frac{(\gamma - 1)}{2}M_\infty{}^2\left(1 - \frac{u^2}{U^2}\right). \tag{40}$$

Using the conditions at the fixed plate namely when $y = 0$, $u = 0$ and $T = T_w$, equation (39) gives

$$\frac{T_w}{T_\infty} = 1 + \frac{P_r Q_w(\gamma - 1)M_\infty{}^2}{\tau_w U} + \frac{P_r(\gamma - 1)M_\infty{}^2}{2}.$$

$$\text{Thus, } Q_w = \frac{\tau_w U}{P_r M_\infty{}^2(\gamma - 1)}\left\{\frac{T_w}{T_\infty} - 1 - \frac{P_r(\gamma - 1)M_\infty{}^2}{2}\right\}. \tag{41}$$

The required velocity distribution is given by equation (23) in the form

$$\tau_w y = \int_0^u \mu\, du. \tag{42}$$

Substituting the value of μ given by equation (29) and the value of T given by equation (41) in equation (39), we get

$$\frac{\tau_w y}{\mu_\infty} = \int_0^u \left\{1 + \frac{P_r Q_w(\gamma - 1)M_\infty{}^2}{\tau_w U}\left(u - \frac{u}{U}\right) + \frac{P_r(\gamma - 1)M_\infty{}^2}{2}\left(1 - \frac{u^2}{U^2}\right)\right\}^m du. \tag{43}$$

For an arbitrary value of m the integral in equation (42) is evaluated numerically. However, if $m = 1$, a simple solution of (42) directly yields

$$\frac{\tau_w y}{\mu_\infty U} = \frac{u}{U} + \frac{P_r Q_w(\gamma - 1)M_\infty{}^2}{\tau_w U}\left\{\frac{u}{U} - \frac{1}{2}\left(\frac{u}{U}\right)^2\right\} + \frac{P_r(\gamma - 1)M_\infty{}^2}{2}\left\{\frac{u}{U} - \frac{1}{3}\left(\frac{u}{U}\right)^3\right\}. \tag{44}$$

Using the boundary condition at the moving plate. namely, when $y = h$, $u = U$, equation (43) reduces to

$$\frac{\tau_w}{\mu_\infty} = 1 + \frac{P_r Q_w(\gamma - 1)M_\infty{}^2}{2\tau_w U} + \frac{P_r(\gamma - 1)M_\infty{}^2}{3}. \tag{45}$$

Solving equations (40) and (44), we obtain values of τ_w and Q_w for a chosen value of y, we obtain u from equation (43). Using the value of u so obtained, we can obtain temperature T, we can obtain μ and ρ with the help of equation (37), the pressure p can be obtained using equation (18).

Adiabatic wall: If the heat transfer at the fixed plate is zero, i.e. if wall is adiabatic, we have $Q_w = 0$. Then, equation (43) reduces to

$$\frac{\tau_w y}{\mu_\infty U} = \frac{u}{U} + \frac{P_r(\gamma - 1)M_\infty{}^2}{2} \left\{ \frac{u}{U} - \frac{1}{3}\left(\frac{u}{U}\right)^3 \right\}. \tag{46}$$

Using the boundary condition at the moving plate, namely, when $y = h$, $u = U$, equation (41) yields

$$\frac{(\tau_w h)}{\mu_\infty U} = 1 + \left(\frac{1}{3}\right) \times P_r M_\infty{}^2(\gamma - 1). \tag{47}$$

Dividing equation (45) by equation (46), we have

$$\frac{Y}{h} = \frac{1}{1 + P_r M_\infty{}^2(\gamma - 1)/3} \left[\frac{u}{U} + \frac{P_r(\gamma - 1)M_\infty{}^2}{2} \left\{ \frac{u}{U} - \frac{1}{3}\left(\frac{u}{U}\right)^3 \right\} \right]. \tag{48}$$

Let $M_\infty \to \infty$. Then equation (47) reduces to

$$\frac{Y}{h} = \left(\frac{3}{2}\right) \times \left(\frac{u}{U}\right) \times \left\{ 1 - \left(\frac{1}{3}\right) \times \left(\frac{u}{U}\right)^2 \right\}. \tag{49}$$

The velocity distribution is plotted against the distance from the fixed plate for various values of $P_r M_\infty{}^2$. From the figure 3, we observe that the effect of the Mach number on the compressible viscous flow is to decrease the velocity gradient at the fixed plate and to increase it at the moving plate since in the above discussion Prandtl number appears paired with Mach number, it has the same influence on the velocity distribution as the Mach number. A comparison of the velocity distribution for $m = 1$ and $m = 0.76$ has been shown in the Figure 3 with $\gamma = 1.4$.

Typical temperature profiles in compressible viscous flow ($P_r M_\infty{}^2 = 2$) for an adiabatic wall ($Q_w = 0$), a heated wall ($Q_w < 0$), and a cooled wall ($Q_w > 0$) are shown in the Figure 4 taking $\gamma = 1.4$. Note that the temperature remains unchanged for an incompressible fluid. Again the temperature gradient become zero for a compressible viscous flow with adiabatic wall. This particular temperature at the fixed insulated wall is known as recovery temperature and is denoted by T_r.

Putting $u = 0$ and $Q_w = 0$ in equation (39), T_r is given by

$$\frac{T_r}{T_\infty} = 1 + P_r M_\infty{}^2(\gamma - 1)/2. \tag{50}$$

The co-efficient of friction at the fixed wall ($Q_w = 0$) can be obtained from equation (46). Thus,

$$C_f = \frac{\tau_w}{\rho_\infty U^2/2} = \frac{1 + P_r M_\infty{}^2(\gamma - 1)/3}{R_e/2}, \tag{51}$$

where R_e = Reynolds number = $\dfrac{Uh}{v_\infty}$ and v_∞ = Kinematic viscosity = $\dfrac{\mu_\infty}{\rho_\infty}$. As shown in Figure 3, the velocity gradient for a compressible viscous fluid varies from the fixed wall to the moving wall and the shearing stress given by equation (46) is constant.

The constancy of the shearing stress can be proved from the velocity gradient given by

$$\tau_w = \mu_w \left(\frac{du}{dy}\right)_w = \mu_\infty \left(\frac{du}{dy}\right)_\infty. \tag{52}$$

In order to prove by equation (42), we have

$$\frac{\mu_w}{\mu_\infty} = \frac{(du/dy)_\infty}{(du/dy)_w}. \tag{53}$$

The viscosity is given by equation (29) and with the help of equation (39), it may be re-written as $(Q_w = 0)$

$$\frac{\mu_w}{\mu_\infty} = \frac{T_w}{T_\infty} = \frac{1 + P_r M_\infty{}^2(\gamma - 1)}{2}. \tag{54}$$

Differentiating equation (45) with respect to y and applying the limits $y = 0$ and $y = h$, we obtain

$$\left(\frac{du}{dy}\right)_w = \frac{T_w}{\mu_\infty} \times + \frac{1}{1 + P_r M_\infty{}^2(\gamma - 1)/2},$$

$$\left(\frac{du}{dy}\right)_w = \frac{\tau_w}{\mu_\infty}. \tag{55}$$

The ratio of the velocity gradient in equation (55) is given by

$$\frac{(du/dy)_\infty}{(du/dy)_w} = 1 + \frac{P_r M_\infty{}^2(\gamma - 1)}{2}. \tag{56}$$

Using equation (56), equation (53) is verified and so the shearing stress is constant in compressible viscous fluid. The increase of the skin-Friction co-efficient at the fixed wall $(Q_w = 0)$ with the increase of the Mach number is illustrated in the Figure 5, taking $\gamma = 1.4$.
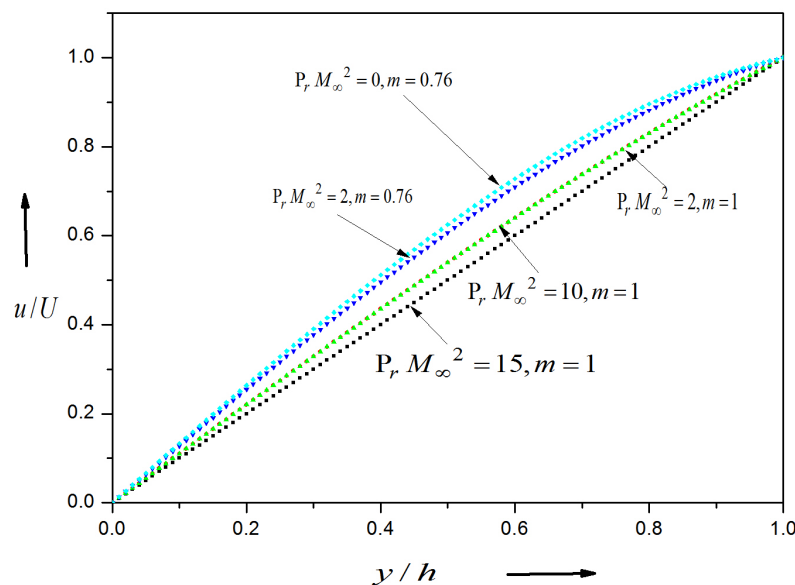
Figure 3: The velocity distribution against the distance from the fixed plate
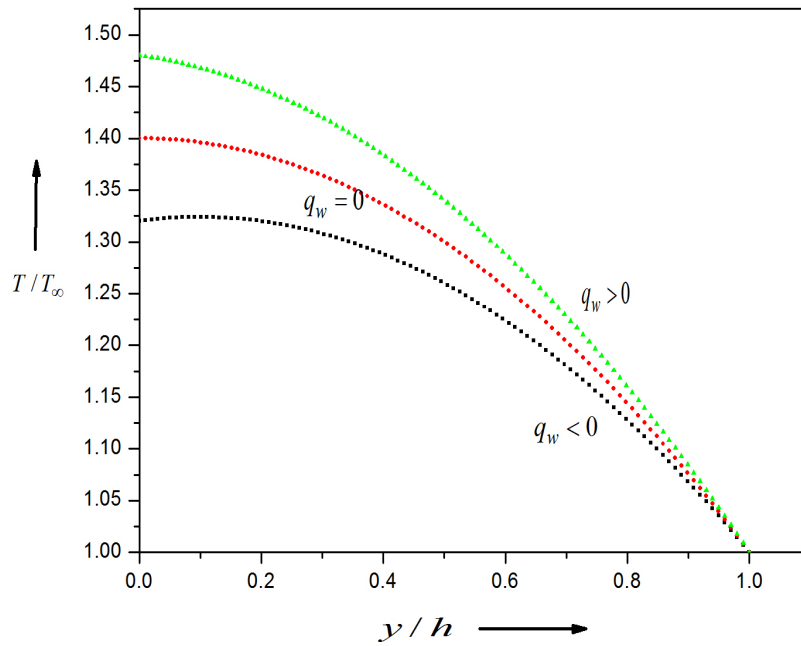
Figure 4: The temperature distribution against the distance from the fixed plate
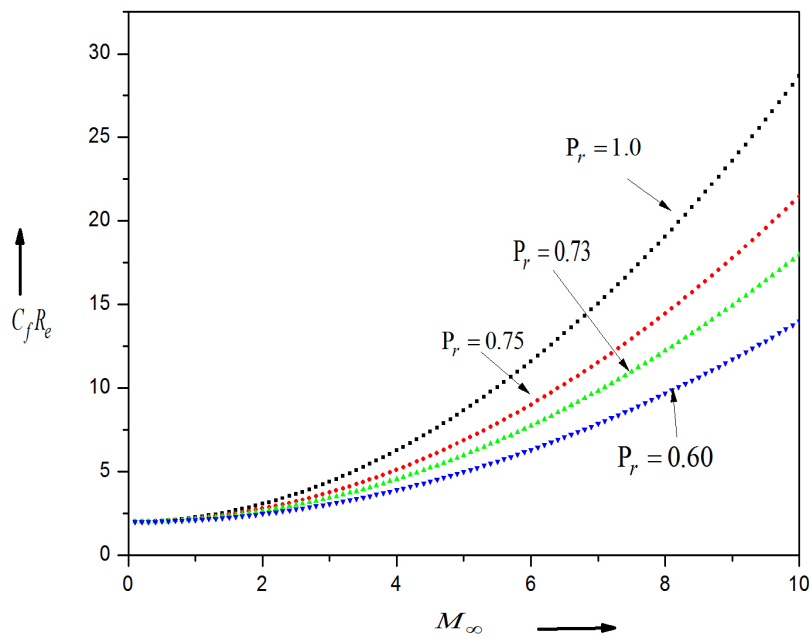
Figure 5: The skin-friction coefficients at the fixed wall ($Q_w = 0$) against Mach number
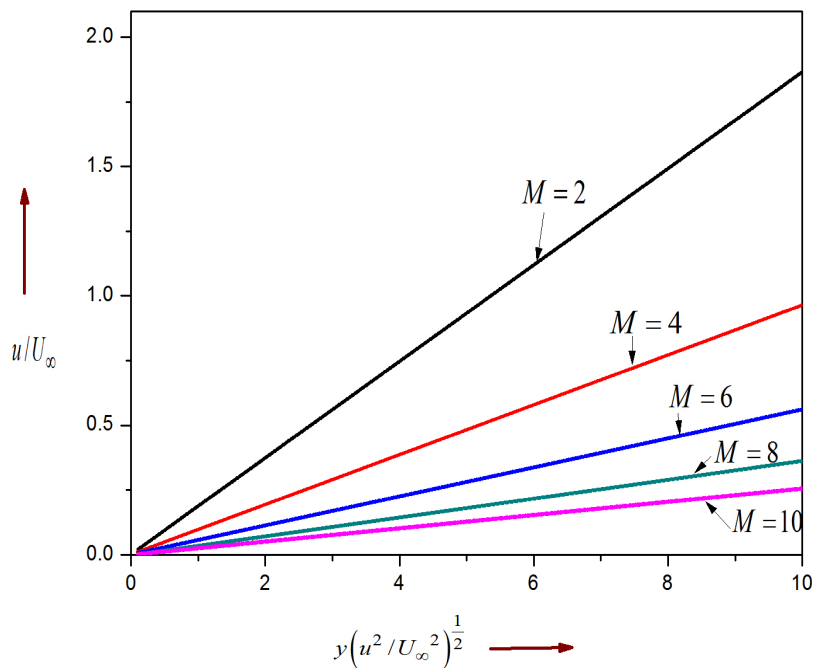
Figure 6: The velocity distribution in the boundary layer
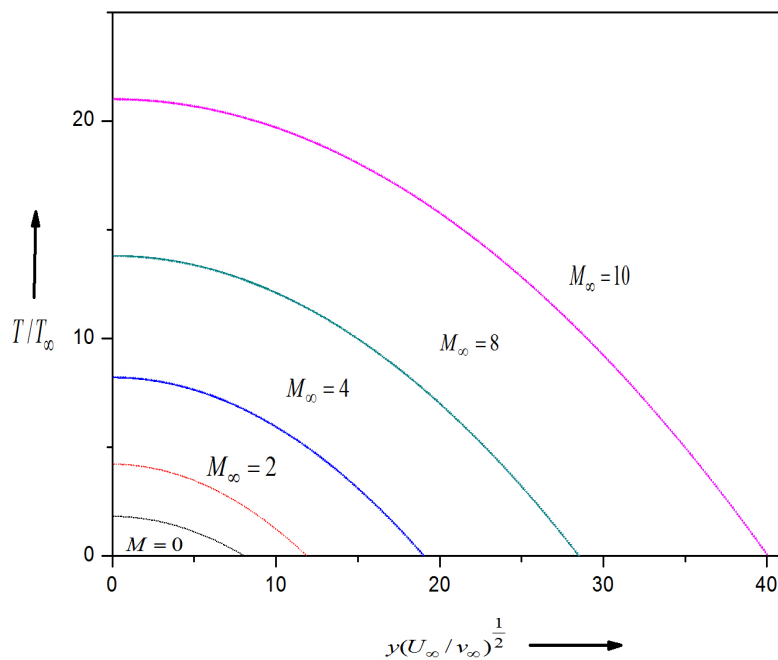
Figure 7: The temperature distribution in the boundary layer

6   Results and Discussions

The velocity distribution is plotted against the distance from the fixed plate for various values of $P_r M_{\text{inf}}^2$ in Figure 3. From the graph in Figure 3 we observe that the effect of Mach number on the compressible viscous flow is to decrease the velocity gradient at the fixed plate and to increase at the moving plate. In graph in Figure 4 temperature profiles for an adiabatic wall $(Q_w = 0)$, a heated wall $(Q_w < 0)$ and a cooled wall $(Q_w > 0)$ is shown. The skin-friction coefficients are illustrated in graph in Figure 5 and also velocity and temperature distribution for Von Kármán-Tsiens's for an insulated plate is given in the graphs in Figures 6 and 7 and hence the velocity and temperature distributions in boundary layer for different Mach number is illustrated.

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Application of 3-scale Haar wavelets to elliptic equations

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Abstract: *Elliptic partial differential equations arise in the mathematical modelling of many physical phenomena arising in science and engineering. In this paper, we obtain the numerical solution of Laplace and Poisson equations using 3-scale Haar wavelets. The basic idea of Haar wavelet collocation method is to convert the partial differential equation into a system of algebraic equations that involves a finite number of variables. The numerical results are compared with the exact solution to prove the accuracy of the Haar wavelet method.*

Keywords: *Elliptic partial differential equations, Laplace equation, Poisson equation, 3-scale Haar wavelets, Collocation points.*

AMS Subject Classification: *65T60.*

1   Introduction

A partial differential equation (PDE) is a mathematical equation that involves two or more independent variables, an unknown functions with respect to the independent variables. All physical PDEs falls into three main categories namely, Parabolic, Hyperbolic and Elliptic. PDEs are used to mathematically formulate, and thus aid the solution of physical and other problems involving functions of several variables, such as the propagation of heat or sound, fluid flow, elasticity, electrostatics, electrodynamics etc.

In the year 1909, Alfred Haar, a Hungarian mathematician introduced Haar function which were later known as Haar wavelets. His contribution to wavelets is very evident. There is an entire wavelet family named after him. The Haar wavelet is a sequence of rescaled "square-shaped" functions which together form a wavelet family or basis. They consist of piecewise constant functions and are therefore the simplest orthonormal wavelets with a compact support. An advantage of these wavelets is the possibility to integrate them analytically for arbitrary times. They are conceptually simple, fast, memory efficient and exactly reversible [1].

In recent years, the wavelet approach for the solution of differential and integral equations has become very popular. Multiresolution analysis of wavelets capture local features efficiently as such enables to detect singularities, shocks, irregular structure and transient phenomena exhibited by the analyzed equations. Chen and Hsiao [2] recommended to expand into the Haar series the highest order derivatives appearing in the differential equation. This idea has been very prolific and it is being abundantly applies for the solution of differential equations. The wavelet coefficients appearing in the Haar series are calculated either using Collocation method or Galerkin method.

Over the recent decades, wavelets by and large have picked up a respectable status because of their applications in different disciplines and in that capacity have many success stories.

Prominent effects of their studies are in the fields of signal processing, computer vision, seismology, turbulence, computer graphics, image processing, structures of the galaxies in the universe, digital communication, pattern recognition, approximation theory, quantum optics, biomedical engineering, sampling theory, matrix theory, operator theory, differential equations, integral equations, numerical analysis, statistics, tomography, and so on. A standout amongst the best utilizations of wavelets has been in image processing. The Federal Bureau of Investigation (FBI) has build up a wavelet based algorithm for fingerprint compression. Wavelets have the capability to designate functions at different levels of resolution, which permits building up a chain of approximate solutions of equations. Compactly supported wavelets are localized in space, wherein solutions can be refined in regions of sharp variations/transients without going for new grid generation, which is the common strategy in classical numerical schemes.

Lepik [3] applied Haar wavelets to solve evolution equations. Shi et al. [4] solved wave equation using Haar wavelets. Ram Jiwari [5] used Haar wavelets to solve Burgers equation. Hariharan et al. [6, 7, 8, 9, 10] applied Haar wavelets to solve Cahn-Allen Equation, Fisher's equation, FitzHugh-Nagumo equation, Klein-Gordon equation, Sine-Gordon equation and some nonlinear parabolic equations. Dhawan et al. [11] solved heat equation using Haar wavelets. Lepik [12] used two-dimensional Haar wavelets to solve diffusion equation and Poisson equation. Wang and Zhao [13] solved two-dimensional Burgers equation using two-dimensional Haar wavelets. Celik [14, 15] used Haar wavelets to solve magnetohydrodynamic flow equations and generalized Burgers-Huxley equation.

Bujurke et al. [16] applied wavelet-multigrid method to solve elliptic partial differential equations. Kumar and Pandit [17] used a composite numerical scheme based on Haar wavelets for the numerical simulation of coupled Burgers equation. Sumana et al. [18, 19, 20] solved two-dimensional hyperbolic, parabolic and elliptic PDEs using two-dimensional 2-scale Haar wavelets. Sumana et al. [21, 22] have also solved two-dimensional Fredholm integral equations and coupled Fredholm integral equations of second kind using two-dimensional 2-scale Haar wavelets. Further, Sumana et. al. [23, 24] obtained the numerical solution of non-homogeneous and non-planar Burgers' equations by 2-scale Haar wavelet method. Kesava et al. [25] used 2-scale Haar wavelets to determine the numerical solution of initial and boundary value problems of ordinary differential equation and system of ordinary differential equations.

Kumar et al. [26] solved Fredholm and Volterra integro differential equations and systems of Fredholm and Volterra integro differential equations numerically using 2-scale Haar wavelets. Savitha et al. [27] obtained the numerical solution of Volterra integral equation and system of Volterra integral equations with the aid of 2-scale Haar wavelets. Tejuswini et al. [28] applied 2-scale Haar wavelets to determine the numerical solution of one-dimensional parabolic PDEs. Mittal and Pandit [29] developed 3-scale Haar wavelets and used it to obtain the numerical solution of planar and non-planar Burgers' equations. Manjunath et al. [30] analyzed the numerical solution of initial and boundary value problems of ordinary differential equations with the help of 3-scale Haar wavelets. Recently, Sumana et al. [31] used 2-scale Haar wavelets to determine the solution of time-delayed Burgers' equations numerically.

In this paper, some elliptic partial differential equations are solved using two-dimensional 3-scale Haar wavelets.

2   Haar Wavelet

The 3-scale Haar wavelets [29] for $x \in [0, 1]$ is defined as follows,

$$h_i(x) = \begin{cases} \psi_i^1(x) & \text{for even } i, \\ \psi_i^2(x) & \text{for odd } i, \end{cases} \tag{1}$$

where

$$\psi_i^1(x) = \frac{1}{\sqrt{2}} \begin{cases} -1 & \text{for } \xi_1 \leq x \leq \xi_2, \\ 2 & \text{for } \xi_2 \leq x \leq \xi_3, \\ -1 & \text{for } \xi_3 \leq x \leq \xi_4, \\ 0 & \text{elsewhere}, \end{cases} \tag{2}$$

$$\psi_i^2(x) = \sqrt{\frac{3}{2}} \begin{cases} 1 & \text{for } \xi_1 \leq x \leq \xi_2, \\ 0 & \text{for } \xi_2 \leq x \leq \xi_3, \\ -1 & \text{for } \xi_3 \leq x \leq \xi_4, \\ 0 & \text{elsewhere}, \end{cases} \tag{3}$$

$$\xi_1 = \frac{k}{m}, \quad \xi_2 = \frac{k + \frac{1}{3}}{m}, \quad \xi_3 = \frac{k + \frac{2}{3}}{m}, \quad \xi_4 = \frac{k+1}{m}. \tag{4}$$

In the above definition $m = 3^j$, $j = 0, 1, \ldots, J$ indicates the level of the wavelet; $k = 0, 1, \ldots m - 1$ is the translation parameter. J is the maximum level of resolution. For index $i = 1$, $h_1(x)$ is assumed to be the scaling function which is defined as follows.

$$h_1(x) = \begin{cases} 1 & \text{for } x \in [0, 1) \\ 0 & \text{elsewhere} \end{cases} \tag{5}$$

For index $i > 1$, even and odd index are calculated from the formula $i = m + 2k + 1$ and $i = m + 2k + 2$ respectively.

In order solve differential equations of any order, we need the following integrals.

$$p_i(x) = \int_0^x h_i(x)dx = \begin{cases} \theta_i^1(x) = \int_0^x \psi_i^1(x)dx & \text{for even } i, \\ \theta_i^2(x) = \int_0^x \psi_i^2(x)dx & \text{for odd } i, \end{cases} \tag{6}$$

where

$$\theta_i^1(x) = \frac{1}{\sqrt{2}} \begin{cases} \xi_1 - x & \text{for } \xi_1 \leq x \leq \xi_2, \\ 2x - 3\xi_2 + \xi_1 & \text{for } \xi_2 \leq x \leq \xi_3, \\ \xi_1 - 3\xi_2 + 3\xi_3 - x & \text{for } \xi_3 \leq x \leq \xi_4, \\ 0 & \text{elsewhere}, \end{cases} \tag{7}$$

$$\theta_i^2(x) = \sqrt{\frac{3}{2}} \begin{cases} x - \xi_1 & \text{for } \xi_1 \leq x \leq \xi_2, \\ \xi_2 - \xi_1 & \text{for } \xi_2 \leq x \leq \xi_3, \\ \xi_3 + \xi_2 - \xi_- x & \text{for } \xi_3 \leq x \leq \xi_4, \\ 0 & \text{elsewhere.} \end{cases} \tag{8}$$

$$q_i(x) = \int_0^x p_i(x)dx = \begin{cases} \zeta^1(x) = \int_0^x \theta_i^1(x)dx & \text{for even } i, \\ \zeta^2(x) = \int_0^x \theta_i^2(x)dx & \text{for odd } i, \end{cases} \tag{9}$$

where

$$\zeta_i^1(x) = \frac{1}{2\sqrt{2}} \begin{cases} -(\xi_1 - x)^2 & \text{for} \xi_1 \leq x \leq \xi_2, \\ 2(x - 2\xi_2 + \xi_1)(x - \xi_2) - (\xi_1 - \xi_2)^2 & \text{for } \xi_2 \leq x \leq \xi_3, \\ (3\xi_3 - 2\xi_2 - x)(x - \xi_3) - (\xi_1 - \xi_2)^2 & \text{for } \xi_3 \leq x \leq \xi_4, \\ 0 & \text{elsewhere,} \end{cases} \tag{10}$$

$$\zeta_i^2(x) = \frac{1}{2}\sqrt{\frac{3}{2}} \begin{cases} (x - \xi_1)^2 & \text{for } \xi_1 \leq x \leq \xi_2, \\ (\xi_2 - \xi_1)(2x - \xi_2 - \xi_1) & \text{for } \xi_2 \leq x \leq \xi_3, \\ \begin{aligned} &(x - \xi_3)(\xi_3 + 2\xi_2 - 2\xi_1 - x) \\ &+ (\xi_2 - \xi_1)(2\xi_3 - \xi_2 - \xi_1) \end{aligned} & \text{for } \xi_3 \leq x \leq \xi_4, \\ \begin{aligned} &(\xi_4 - \xi_3)(\xi_3 + 2\xi_2 - 2\xi_1 - \xi_4) \\ &+ (\xi_2 - \xi_1)(2\xi_3 - \xi_2 - \xi_1) \end{aligned} & \text{for } \xi_4 \leq x \leq 1, \\ 0 & \text{elsewhere.} \end{cases} \tag{11}$$

The Haar wavelets (1)-(5) and its integrals (6)-(11) for $i = 1, 2, \ldots, 9$ are presented in Figures 1-3.

2.1   Function approximation

Any function $g(x, y)$ which is square integrable on $[0, 1) \times [0, 1)$ can be expressed as an infinite sum of Haar wavelets as

$$g(x, y) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} b_{ij} h_i(x) h_j(y), \tag{12}$$

$$b_{ij} = \int_0^1 \int_0^1 g(x, y) h_i(x) h_j(y) dx \, dy. \tag{13}$$

If $g(x, y)$ is approximated as piecewise constant in each sub-area, then equation (12) will be terminated at finite terms, i.e.

$$g(x, y) = \sum_{i=1}^{3M_1} \sum_{j=1}^{3M_2} b_{ij} h_i(x) h_j(y), \tag{14}$$

where the wavelet coefficients b_{ij}, $i = 1, 2, \ldots, 3M_1$, $j = 1, 2, \ldots, 3M_2$ are to be determined. Here $M_1 = 3^{J_1}$ and $M_2 = 3^{J_2}$, and J_1, J_2 are the maximum levels of the resolution of the wavelet.
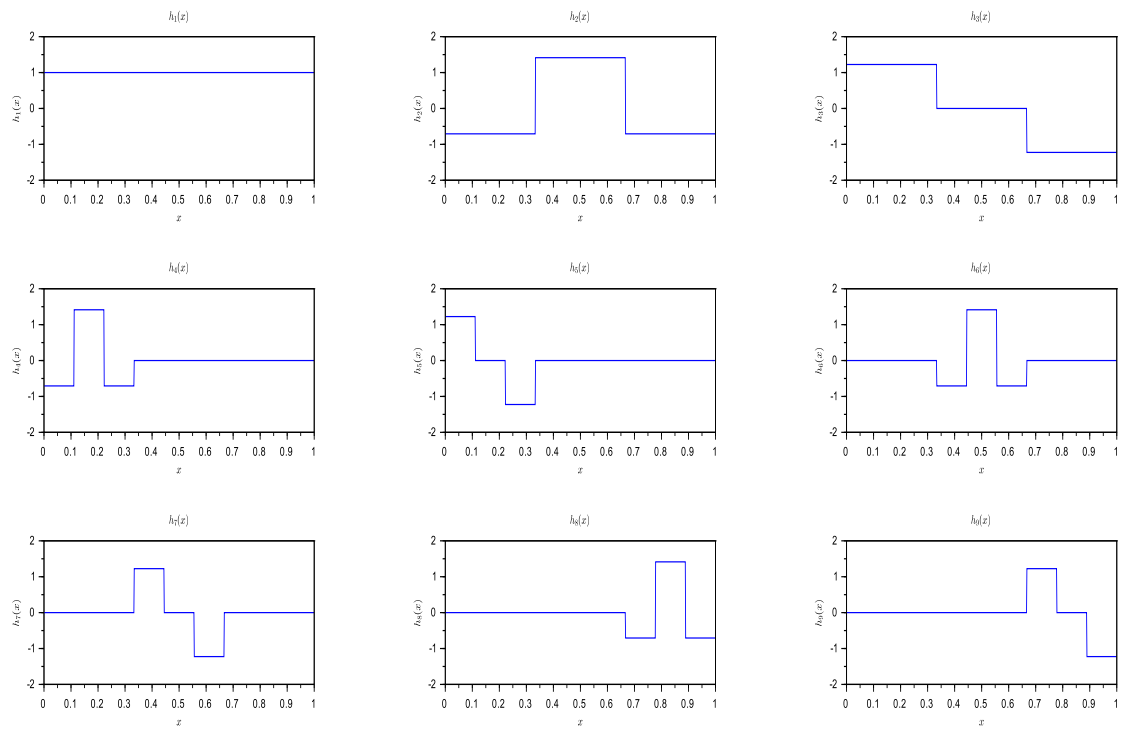
Figure 1: $h_i(x)$ for $i = 1, 2, \ldots, 9$
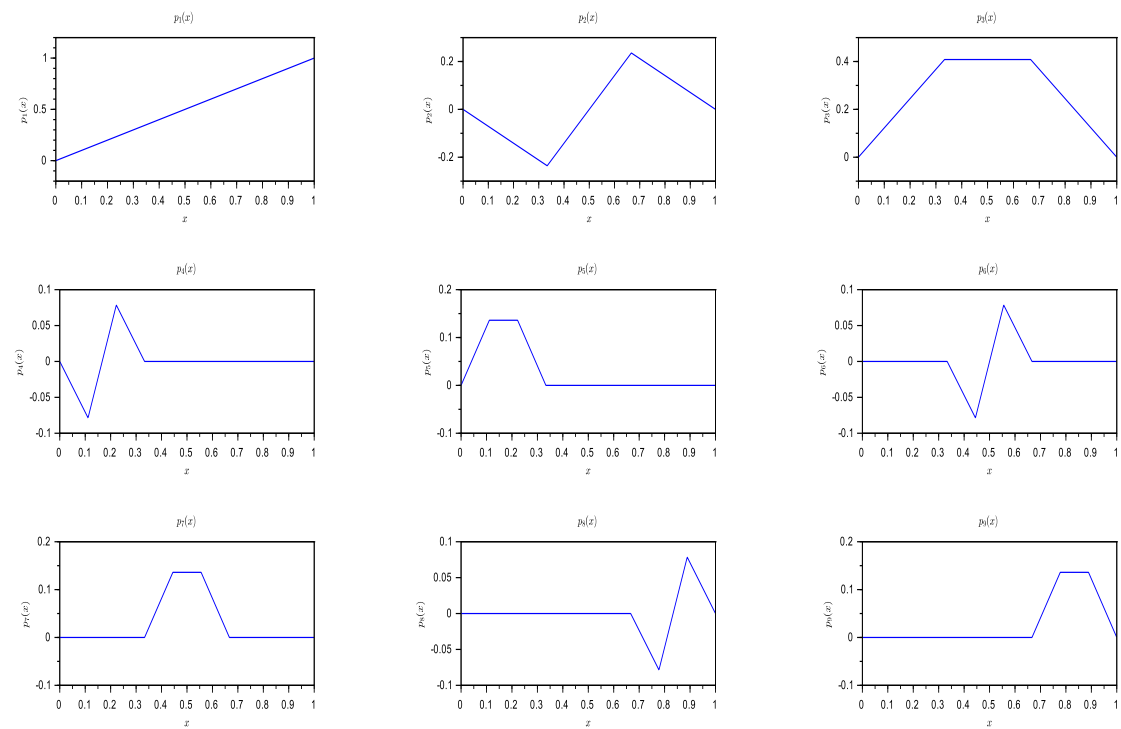
Figure 2: $p_i(x)$ for $i = 1, 2, \ldots, 9$
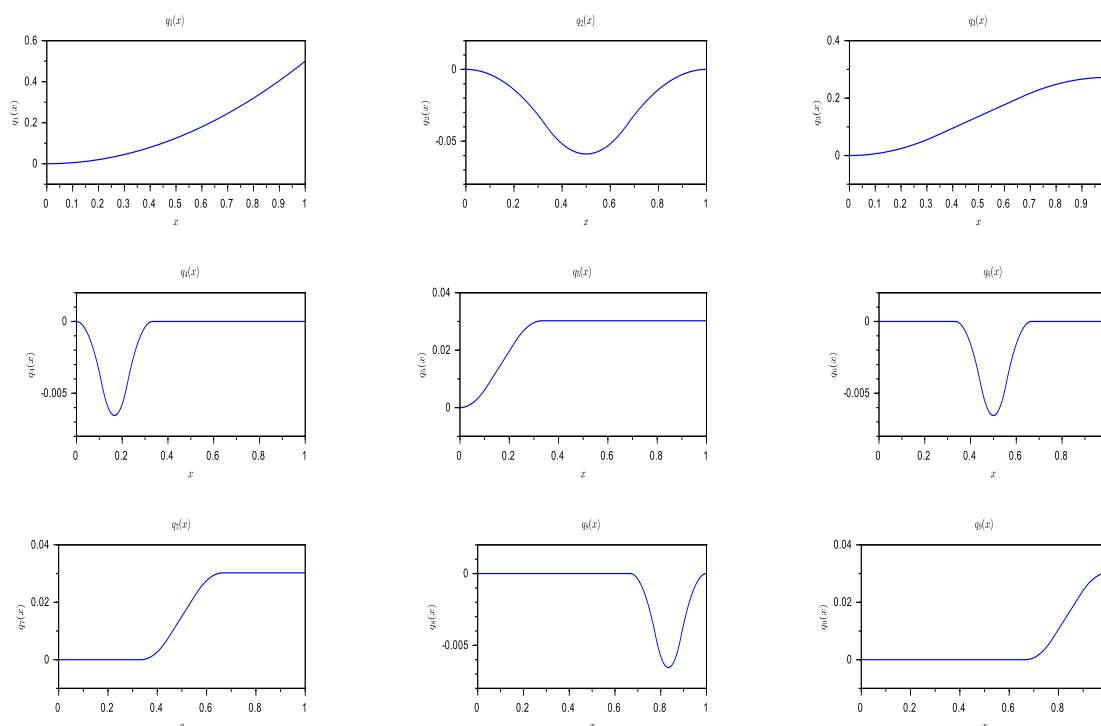
Figure 3: $q_i(x)$ for $i = 1, 2, \ldots, 9$

3   Method of Solution

In this section, the description of the Haar wavelet collocation method (HWCM) to solve two-dimensional elliptic PDEs is outlined.

3.1   Laplace Equation

Consider the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \ 0 \leq x \leq 1, \ 0 \leq y \leq 1, \tag{15}$$

with boundary conditions

$$u(x, 0) = f_1(x), 0 \leq x \leq 1, \tag{16}$$

$$u(x, 1) = f_2(x), \ 0 \leq x \leq 1, \tag{17}$$

$$u(0, y) = g_1(y), \ 0 \leq y \leq 1, \tag{18}$$

$$u(1, y) = g_2(y), \ 0 \leq y \leq 1. \tag{19}$$

The order of the PDE (15) is 2 w.r.t. x and 2 w.r.t. y. Therefore the Haar wavelet solution is assumed to be in the form

$$u_{xxyy}(x, y) = \sum_{i=1}^{3M_1} \sum_{j=1}^{3M_2} a(i, j) h_i(x) h_j(y). \tag{20}$$

Integrating equation (20) twice w.r.t y in the limits $[0, y]$ and using equation (16) gives

$$u_{xx}(x, y) = \sum_{i=1}^{3M_1} \sum_{j=1}^{3M_2} a(i, j) h_i(x) q_j(y) + y u_{xxy}(x, 0) + f_1''(x). \tag{21}$$

Putting $y = 1$ in equation (21) and using equation (17) leads to

$$u_{xxy}(x, 0) = -\sum_{i=1}^{3M_1} \sum_{j=1}^{3M_2} a(i, j) h_i(x) q_j(1) + f_2''(x) - f_1''(x). \tag{22}$$

Substituting equation (22) in equation (21), we obtain

$$u_{xx}(x, y) = \sum_{i=1}^{3M_1} \sum_{j=1}^{3M_2} a(i, j) h_i(x) [q_j(y) - y q_j(1)] + y f_2''(x) + (1 - y) f_1''(x). \tag{23}$$

Integrating equation (20) twice w.r.t. x in the limits $[0, x]$ and using equation (18), we arrive at

$$u_{yy}(x, y) = \sum_{i=1}^{3M_1} \sum_{j=1}^{3M_2} a(i, j) q_i(x) h_j(y) + x u_{xxy}(0, y) + g_1''(y). \tag{24}$$

Putting $x = 1$ in equation (24) and using (19), we have

$$u_{xyy}(0, y) = -\sum_{i=1}^{3M_1} \sum_{j=1}^{3M_2} a(i, j) q_i(1) h_j(y) + g_2''(y) - g_1''(y). \tag{25}$$

Substituting equation (25) in equation (24), we get

$$u_{yy}(x, y) = \sum_{i=1}^{3M_1} \sum_{j=1}^{3M_2} a(i, j) [q_i(x) - x q_i(1)] h_j(y) + x g_2''(y) + (1 - x) g_1''(y). \tag{26}$$

Integrating equation (23) twice w.r.t. x in the limits $[0, x]$ and using equation (18), gives

$$u(x, y) = \sum_{i=1}^{3M_1} \sum_{j=1}^{3M_2} a(i, j) q_i(x) [q_j(y) - y q_j(1)] + x u_x(0, y) + g_1(y) + y f_2(x)$$
$$+ (1 - y) f_1(x) - y[f_2(0) + x f_2'(0)] - (1 - y)[f_1(0) + x f_1'(0)]. \tag{27}$$

Putting $x = 1$ in equation (27) and using equation (19) leads to

$$u_x(0, y) = -\sum_{i=1}^{3M_1} \sum_{j=1}^{3M_2} a(i, j) q_i(1) [q_j(y) - y q_j(1)] + g_2(y) - g_1(y)$$
$$+ y[f_2(0) + f_2'(0) - f_2'(1)] - (1 - y)[f_1(0) + x f_1'(0) - f_1'(1)]. \tag{28}$$

Substituting equation (28) in equation (27), we obtain

$$u(x, y) = \sum_{i=1}^{3M_1} \sum_{j=1}^{3M_2} a(i, j) [q_i(x) - x q_i(1)] + x g_2(y) + (1 - x) g_1(y) + y f_2(x)$$
$$+ (1 - y) f_1(x) - x y f_2(1) - x(1 - y) f_1(1) - (1 - x) y f_2(0)$$
$$- (1 - x)(1 - y) f_1(0). \tag{29}$$

The wavelet collocation points are defined as

$$x_l = \frac{l - 0.5}{3M_1}, \quad l = 1, 2, 3, \ldots, 3M_1, \tag{30}$$

$$y_n = \frac{n - 0.5}{3M_2}, \quad n = 1, 2, 3, \ldots, 3M_2, \tag{31}$$

Substituting equations (23) and (26) in equation (15), and taking $x \to x_l$ and $y \to y_n$ in the resultant equations and equation (29), we get

$$\sum_{i=1}^{2M_1} \sum_{j=1}^{2M_2} a(i,j) A(i,j,l,n) = \phi(x_l, y_n), \tag{32}$$

where

$$A(i,j,l,n) = h_i(x_l)[q_j(y_n) - y_n q_j(1)] + [q_i(x_l) - x_l q_i(1)] h_j(y_n), \tag{33}$$

$$\phi(x_l, y_n) = (y_n - 1) f_1''(x_l) - y_n f_2''(x_l) + (x_l - 1) g_1''(y_n) - x_l g_2''(y_n), \tag{34}$$

$$\begin{aligned}
u(x_l, y_n) = {} & \sum_{i=1}^{2M_1} \sum_{j=1}^{2M_2} a(i,j)[q_i(x_l) - x_l q_i(1)][q_j(y_n) - y_n q_j(1)] + x_l g_2(y_n) \\
& + (1 - x_l) g_1(y_n) + y_n f_2(x_l) + (1 - y_n) f_1(x_l) - x_l y_n f_2(1) \\
& - x_l(1 - y_n) f_1(1) - (1 - x_l) y_n f_2(0) - (1 - x_l)(1 - y_n) f_1(0).
\end{aligned} \tag{35}$$

In order to calculate the wavelet coefficients $a(i,j)$ from equation (32), we must transform the system into a form containing only two-order matrices as there are no algorithms to deal with fourth-order matrices. We define new indices,

$$\alpha = 3M_1(i - 1) + j, \ \beta = 3M_2(l - 1) + n. \tag{36}$$

We rewrite the equation (32) in the form

$$\sum_{i=1}^{3M_1} \sum_{j=1}^{3M_2} b(\alpha) R(\alpha, \beta) = \psi(\beta), \tag{37}$$

where $b(\alpha) \to a(i,j), \psi(\beta) \to \phi(x_l, y_n)$ and $R(\alpha, \beta) \to A(i,j,l,n)$. Now b and ψ are $3M_1 * 3M_2$ dimensional row vectors and R is a $(3M_1)^2 * (3M_2)^2$ dimensional matrix. We first evaluate $b(\alpha)$ from equation (37) and then re-establish the wavelet coefficients $a(i,j)$ using (36).

$$\frac{\alpha}{3M_1} = i - 1 + \frac{l}{3M_1}, \frac{\beta}{3M_2} = i - 1 + \frac{n}{3M_2}. \tag{38}$$

The integer part of equation (38) gives $i - 1$ and $l - 1$, and the remainders are j and n. In this way, the wavelet coefficients $a(i,j); i = 1, 2, \ldots 3M_1, \ j = 1, 2, \ldots 3M_2$ can be calculated from equation (32). These coefficients are then substituted in equation (35) to obtain the Haar wavelet solution at the collocation points $x_l, l = 1, 2, \ldots 3M_1, y_n, n = 1, 2, \ldots 3M_2$.

The Laplace equation and Poisson equation have the highest order of the derivative w.r.t. x and y as 2 and 2 respectively. Therefore, the Haar wavelet method outlined in (20)-(29) is common to both the equations.

3.2 Poisson Equation

Consider the Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F(x, y), \quad 0 \le x \le 1,\ 0 \le y \le 1, \tag{39}$$

with boundary conditions

$$u(x, 0) = f_1(x),\ 0 \le x \le 1, \tag{40}$$

$$u(x, 1) = f_2(x),\ 0 \le x \le 1, \tag{41}$$

$$u(0, y) = g_1(y),\ 0 \le y \le 1, \tag{42}$$

$$u(1, y) = g_2(y),\ 0 \le y \le 1, \tag{43}$$

where $F(x, y)$ is the inhomogeneous term.

Substituting equations (23) and (26) in equation (39), and taking $x \to x_l$ and $y \to y_n$ in the resultant equations, we get

$$\sum_{i=1}^{2M_1} \sum_{j=1}^{2M_2} a(i, j) A(i, j, l, n) = \phi(x_l, y_n), \tag{44}$$

where

$$A(i, j, l, n) = h_i(x_l)[q_j(y_n) - y_n q_j(1)] + [q_i(x_l) - x_l q_i(1)] h_j(y_n), \tag{45}$$

$$\phi(x_l, y_n) = (y_n - 1) f_1''(x_l) - y_n f_2''(x_l) + (x_l - 1) g_1''(y_n) - x_l g_2''(y_n), \tag{46}$$

In order to calculate the approximate solution of the Poisson equation (39), the wavelet coefficients $a(i, j)$, $i = 1, 2, \ldots, 3M_1$, $j = 1, 2, \ldots, 3M_2$ computed from equation (44) are substituted in equation (35).

Error Estimate: We define the error estimate at $t = t_s$ by

$$\nu(t_s) = \frac{1}{3M_1} \frac{1}{3M_2} \| u(x, y, t_s) - u_{ex}(x, y, t_s) \|, \tag{47}$$

where $u_{ex}(x, y, t_s)$ is the exact solution at $t = t_s$.

4 Examples and Discussions

In this section, two examples each of Laplace equation and Poisson equation are discussed. Lagrange bivariate interpolation is used to determine the solution at the specified points. The entire computational work is done using MATLAB.

Example 1:

Consider the two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \le x \le 1,\ 0 \le y \le 1, \tag{48}$$

subject to the boundary conditions

$$u(x, 0) = 0, \ 0 \leq x \leq 1, \tag{49}$$

$$u(x, 1) = \sin(\pi x), \ 0 \leq x \leq 1, \tag{50}$$

$$u(0, y) = 0, \ 0 \leq y \leq 1, \tag{51}$$

$$u(1, y) = 0, \ 0 \leq y \leq 1. \tag{52}$$

The exact solution is

$$u(x, y) = \frac{\sin(\pi x) \sinh(\pi y)}{\sinh(\pi)}. \tag{53}$$

The HWCM solution of this example with $J_1 = J_2 = 2$ is given in Table 1. The results are compared with the exact solution and are found to be in good agreement. Figure 4 shows the comparison of the HWCM solution with the exact solution. The error estimates obtained for different J_1, J_2 are given in Table 2.

Example 2:

Consider the two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 1 \leq x \leq 2, \ 0 \leq y \leq 1, \tag{54}$$

subject to the boundary conditions

$$u(x, 0) = 2\log(x), \ 1 \leq x \leq 2, \tag{55}$$

$$u(x, 1) = \log(x^2 + 1), \ 1 \leq x \leq 2, \tag{56}$$

$$u(0, y) = \log(y^2 + 1), \ 0 \leq y \leq 1, \tag{57}$$

$$u(1, y) = \log(y^2 + 4), \ 0 \leq y \leq 1. \tag{58}$$

The exact solution is

$$u(x, y) = \log(x^2 + y^2). \tag{59}$$

The HWCM solution of this example with $J_1 = J_2 = 2$ is given in Table 3. The results are compared with the exact solution and are found to be in good agreement. Figure 5 shows the comparison of the HWCM solution with the exact solution. The error estimates obtained for different J_1, J_2 are given in Table 4.

Example 3:

Consider the two-dimensional Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (x^2 + y^2), \quad 0 \leq x \leq 1, \ 0 \leq y \leq 1, \tag{60}$$

subject to the boundary conditions

$$u(x, 0) = 1, \ 0 \leq x \leq 1, \tag{61}$$

$$u(x, 1) = e^x, \ 0 \leq x \leq 1, \tag{62}$$

$$u(0, y) = 1, \ 0 \leq y \leq 1, \tag{63}$$

$$u(1, y) = e^y, \ 0 \leq y \leq 1. \tag{64}$$

The exact solution is

$$u(x, y) = e^{xy}. \tag{65}$$

The HWCM solution of this example with $J_1 = J_2 = 2$ is given in Table 5. The results are compared with the exact solution and are found to be in good agreement. Figure 6 shows the comparison of the HWCM solution with the exact solution. The error estimates obtained for different J_1, J_2 are given in Table 6.

Example 4:

Consider the two-dimensional Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{x}{y} + \frac{y}{x}, \quad 1 \leq x \leq 2, \ 1 \leq y \leq 2, \tag{66}$$

subject to the boundary conditions

$$u(x, 0) = x \log(x), \ 1 \leq x \leq 2, \tag{67}$$

$$u(x, 1) = x \log(4x^2), \ 1 \leq x \leq 2, \tag{68}$$

$$u(0, y) = y \log(y), \ 1 \leq y \leq 2, \tag{69}$$

$$u(1, y) = y \log(4y^2), \ 1 \leq y \leq 2. \tag{70}$$

The exact solution is

$$u(x, y) = xy \log(xy). \tag{71}$$

The HWCM solution of this example with $J_1 = J_2 = 2$ is given in Table 7. The results are compared with the exact solution and are found to be in good agreement. Figure 7 shows the comparison of the HWCM solution with the exact solution. The error estimates obtained for different J_1, J_2 are given in Table 8.

5   Conclusion

In this paper, an efficient numerical scheme based on uniform 3-scale Haar wavelets is used to solve elliptic partial differential equations, namely, two-dimensional Laplace and Poisson equations. The numerical scheme is tested for four examples. The obtained numerical results are compared with the exact solutions. We observe that the error values are negligibly small which indicate that the HWCM solution is very close to the exact solution. Thus the 3-scale Haar wavelet method guarantees the necessary accuracy with a small number of grid points and a wide class of PDEs can be solved using this approach. This method takes care of boundary conditions automatically and hence it is the most convenient method for solving boundary value problems. This method can also be used to solve nonlinear PDEs.

Table 1: Comparison of the HWCM solution and exact solution of Example 1

(x,y)	$u(x,y)$		(x,y)	$u(x,y)$	
	HWCM	Exact		HCWM	Exact
(0.1,0.2)	0.0179784384	0.0179405685	(0.5,0.6)	0.2785441683	0.2785683338
(0.1,0.4)	0.0432409941	0.0431998876	(0.5,0.8)	0.5307515912	0.5309792250
(0.1,0.6)	0.0860748486	0.0860823492	(0.7,0.2)	0.0470681808	0.0469690182
(0.1,0.8)	0.1640111984	0.1640816042	(0.7,0.4)	0.1132064354	0.1130987739
(0.3,0.2)	0.0470681808	0.0469690182	(0.7,0.6)	0.2253469651	0.2253665161
(0.3,0.4)	0.1132064354	0.1130987739	(0.7,0.8)	0.4293870555	0.4295712167
(0.3,0.6)	0.2253469651	0.2253665161	(0.9,0.2)	0.0179784384	0.0179405685
(0.3,0.8)	0.4293870555	0.4295712167	(0.9,0.4)	0.0432409941	0.0431998876
(0.5,0.2)	0.0581794712	0.0580568993	(0.9,0.6)	0.0860748486	0.0860823492
(0.5,0.4)	0.1399308501	0.1397977728	(0.9,0.8)	0.1640111984	0.1640816042

Table 2: Error in the solution of Example 1

J_1	J_2	$\nu(t)$	
		L_2	L_∞
1	1	1.0247E-05	1.2877E-05
2	2	4.4110E-07	5.3499E-07
3	3	1.6605E-08	2.0086E-08

Table 3: Comparison of the HWCM solution and exact solution of Example 2

(x,y)	$u(x,y)$		(x,y)	$u(x,y)$	
	HWCM	Exact		HCWM	Exact
(0.1,0.2)	0.2231332932	0.2231435513	(0.5 ,0.6)	0.9593639941	0.9593502213
(0.1 ,0.4)	0.3148025321	0.3148107398	(0.5 ,0.8)	1.0612695990	1.0612565021
(0.1 ,0.6)	0.4510703158	0.4510756194	(0.7 ,0.2)	1.0749916546	1.0750024230
(0.1 ,0.8)	0.6151830671	0.6151856391	(0.7 ,0.4)	1.1151437271	1.1151415906
(0.3 ,0.2)	0.5481025156	0.5481214085	(0.7 ,0.6)	1.1786684372	1.1786549963
(0.3 ,0.4)	0.6151785570	0.6151856391	(0.7 ,0.8)	1.2613100773	1.2612978709
(0.3 ,0.6)	0.7178435591	0.7178397932	(0.9 ,0.2)	1.2947236233	1.2947271676
(0.3 ,0.8)	0.8458736204	0.8458682676	(0.9 ,0.4)	1.3270762163	1.3270750015
(0.5 ,0.2)	0.8285349823	0.8285518176	(0.9 ,0.6)	1.3787714780	1.3787660947
(0.5 ,0.4)	0.8796261923	0.8796267475	(0.9 ,0.8)	1.4469238034	1.4469189829

Table 4: Error in the solution of Example 2

J_1	J_2	$\nu(t)$	
		L_2	L_∞
1	1	8.4940E-07	1.1409E-06
2	2	3.4791E-08	4.6102E-08
3	3	1.3030E-09	1.7234E-09

Table 5: Comparison of the HWCM solution and exact solution of Example 3

(x, y)	$u(x, y)$		(x, y)	$u(x, y)$	
	HWCM	Exact		HCWM	Exact
(0.1 ,0.2)	1.0202091485	1.0202013400	(0.5 ,0.6)	1.3499088159	1.3498588076
(0.1 ,0.4)	1.0408238381	1.0408107742	(0.5 ,0.8)	1.4918611920	1.4918246976
(0.1 ,0.6)	1.0618510247	1.0618365465	(0.7 ,0.2)	1.1502986725	1.1502737989
(0.1 ,0.8)	1.0832974860	1.0832870677	(0.7 ,0.4)	1.3231714017	1.3231298123
(0.3 ,0.2)	1.0618569566	1.0618365465	(0.7 ,0.6)	1.5220084826	1.5219615556
(0.3 ,0.4)	1.1275308401	1.1274968516	(0.7 ,0.8)	1.7507074836	1.7506725003
(0.3 ,0.6)	1.1972549731	1.1972173631	(0.9 ,0.2)	1.1972284352	1.1972173631
(0.3 ,0.8)	1.2712762642	1.2712491503	(0.9 ,0.4)	1.4333480852	1.4333294146
(0.5 ,0.2)	1.1051979068	1.1051709181	(0.9 ,0.6)	1.7160283138	1.7160068622
(0.5 ,0.4)	1.2214476803	1.2214027582	(0.9 ,0.8)	2.0544496553	2.0544332106

Table 6: Error in the solution of Example 3

J_1	J_2	$\nu(t)$	
		L_2	L_∞
1	1	2.9651E-06	3.7196E-06
2	2	1.1237E-07	1.4133E-07
3	3	4.1726E-09	5.24492E-09

Table 7: Comparison of the HWCM solution and exact solution of Example 4

(x, y)	$u(x, y)$		(x, y)	$u(x, y)$	
	HWCM	Exact		HCWM	Exact
(0.1 ,0.2)	0.3664739017	0.3664738923	(0.5 ,0.6)	2.1011249698	2.1011249696
(0.1 ,0.4)	0.6649449341	0.6649449213	(0.5 ,0.8)	2.6817797888	2.6817797871
(0.1 ,0.6)	0.9949523188	0.9949523039	(0.7 ,0.2)	1.4544176050	1.4544176080
(0.1 ,0.8)	1.3525317704	1.3525317525	(0.7 ,0.4)	2.0636991601	2.0636991607
(0.3 ,0.2)	0.6937098796	0.6937098812	(0.7 ,0.6)	2.7217187141	2.7217187144
(0.3 ,0.4)	1.0898824324	1.0898824320	(0.7 ,0.8)	3.4223496442	3.4223496429
(0.3 ,0.6)	1.5233252197	1.5233252189	(0.9 ,0.2)	1.8791199998	1.8791200100
(0.3 ,0.8)	1.9893531769	1.9893531747	(0.9 ,0.4)	2.6023474780	2.6023474866
(0.5 ,0.2)	1.0580159945	1.0580159968	(0.9 ,0.6)	3.3800468375	3.3800468469
(0.5 ,0.4)	1.5580684238	1.5580684239	(0.9 ,0.8)	4.2053706760	4.2053706847

Table 8: Error in the solution of Example 4

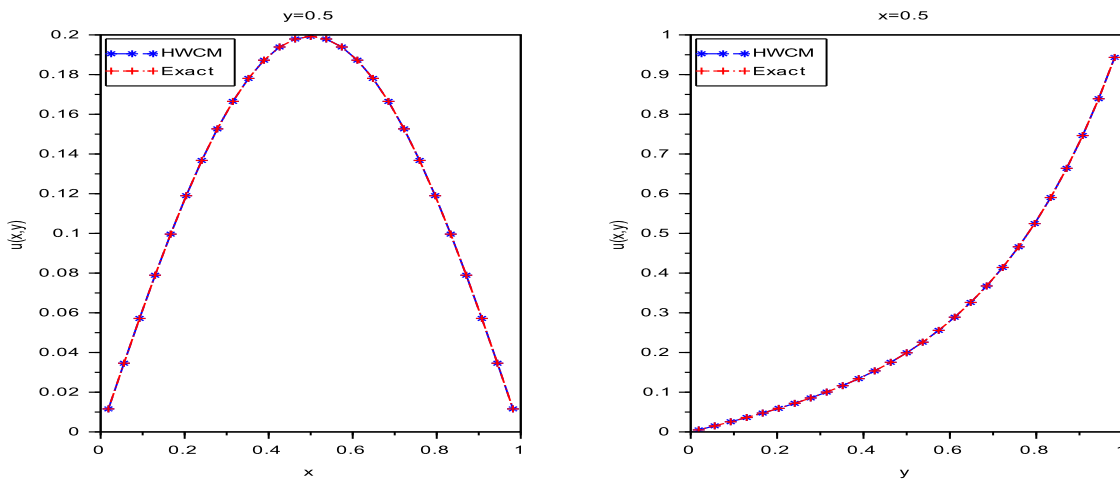| J_1	J_2	$\nu(t)$	
		L_2	L_∞
1	1	2.4892E-17	4.9343E-17
2	2	9.6869E-18	2.4671E-17
3	3	2.3485E-18	8.4269E-18

Figure 4: Comparison of the HWCM solution and exact solution of Example 1
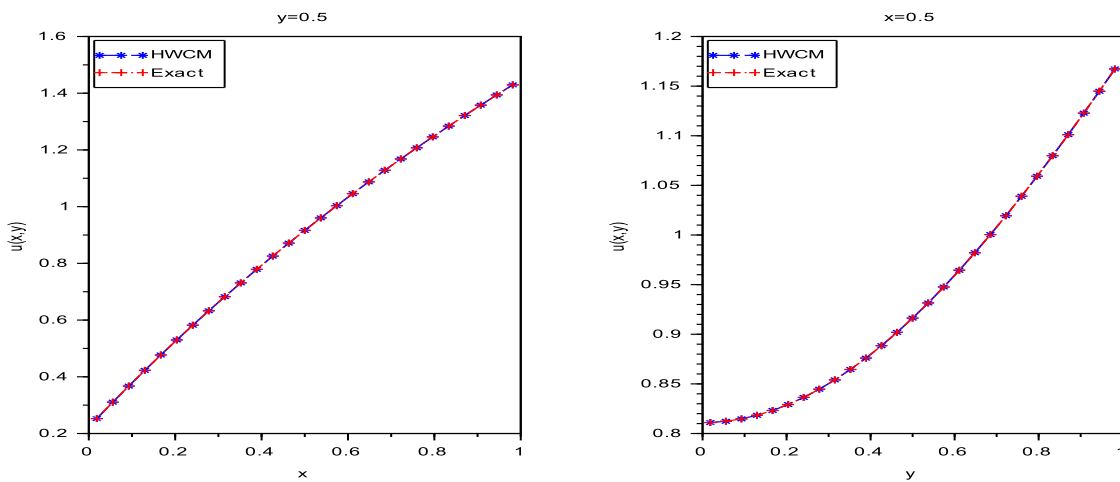

Figure 5: Comparison of the HWCM solution and exact solution of Example 2
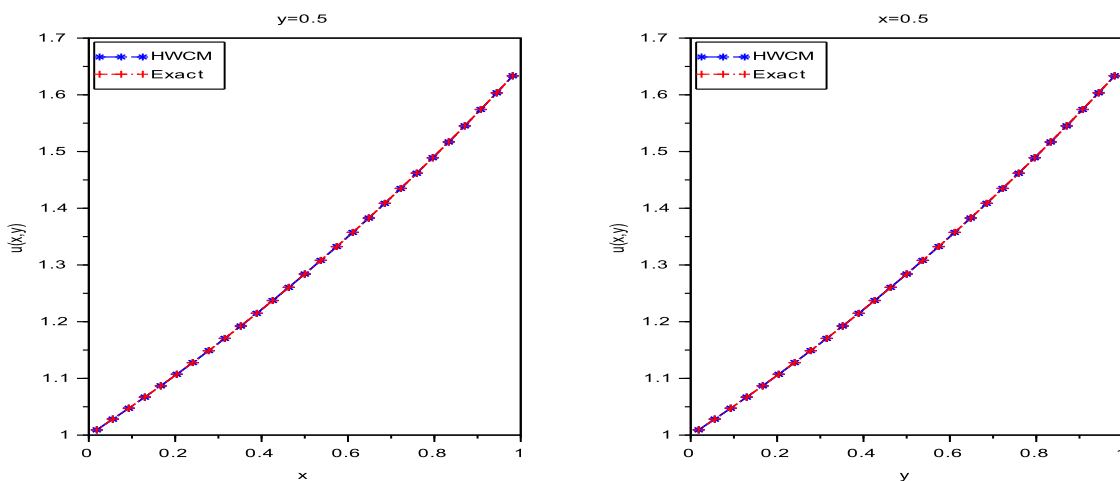
Figure 6: Comparison of the HWCM solution and exact solution of Example 3
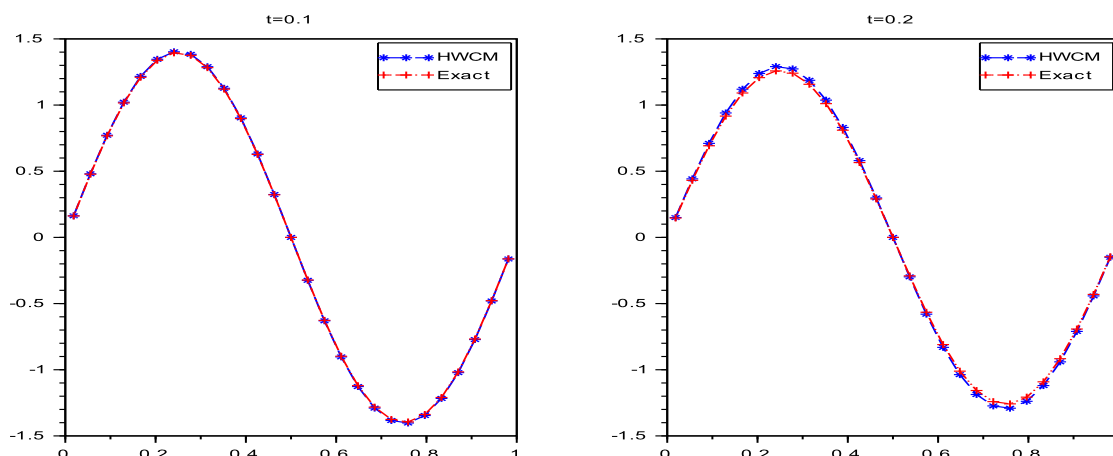
Figure 7: Comparison of the HWCM solution and exact solution of Example 4

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Magnetic fluid flow between two parallel plates with constant pressure gradient

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Abstract: *The nature of ferrofluid with constant pressure gradient is analysed between parallel plates. Maximum velocity, Average velocity and mass flow rate are determined.*

Keywords: *Ferrofluids,Couette flow, maximum velocity, average velocity and mass flow rate.*

AMS Subject Classification: *76DXX.*

1   Introduction

Magnetic fluids are also known as Ferrofluids and are special category of smart nano materials, in particular they are magnetically controllable nano fluids. Ferrofluid was invented in 1963 by NASA's Steven Papell as liquid rocket fuel that could be drawn towards a pump inlet in a weightless environment by applying magnetic field. This types of nanofluids are colloids of magnetic nano particles, such as Fe_3o_4, Co, Fe etc, Stably dispersed in a carrier liquid. Consequently, these nano materials manifest simultaneously fluid and magnetic properties. Macroscopically, the introduction of magnetic force into the fundamental hydrodynamic equation for the quasihomogeneous magnetisable liquid medium gives rise to the magnetohydrodynamics of magnetic nanofluids (Ferrofluids).

Some art and science museums have special devices on display that use magnets to make magnetic fluids move around specially shaped surfaces in a fountain show-like fashion to entering guests. Sachiko Kodama is known for her magnetic fluid art. The Australian electronic rock band Pendulum used magnetic fluid for the music video for track, Watercolour. The design house Krafted London was responsible for the magnetic fluid FX in the video. The post-metal band lsis also used a magnetic fluid in the music-video for 20 minutes/40 years. Martin Frey, a German designer and technologist, built the pixel-based ferrofluid display Sn Oil in 2005. The device consists of a 12×12 matrix of electromagnetics, allowing the apparatus to display text and run simple games.

Magnetic fluids have been proposed for magnetic drug targetting. In this process the drugs would be attached to or enclosed within a magnetic fluid and could be targeted and selectively released using magnetic fields. They are used as "Tag and Drag" removal of toxins. It has also been proposed for targeted magnetic hyperthermia to convert electromagnetic energy into heat. It has also been proposed in a form of nano-surgery to separate one tissue from another for example a tumour from the tissue in which it has grown [1].

Verma [2] showed that the non linearities of the governing equations of the ferrofluids will cancel for constant Laplacian and vorticity on stream lines. They found an angle between the magnetization and the applied magnetic field and determined the exact solution. Achala et al.

[3] examined both polar and nonpolar ferrofluid between two parallel plates. They found the velocity and shear stress for both polar and nonpolar ferrofluid and obtained the solution of the spin velocity of polar fluid.

2   Geometry and Governing equations

Consider a steady laminar flow of an incompressible viscous fluid between two parallel rigid plates separated by a distance h due to the moment of the upper plate with uniform velocity U_i while the lower plate is stationary and constant pressure gradient Such a flow is called Generalized plane Couette flow. Let x be the direction of the flow, z be the direction perpendicular to the flow.
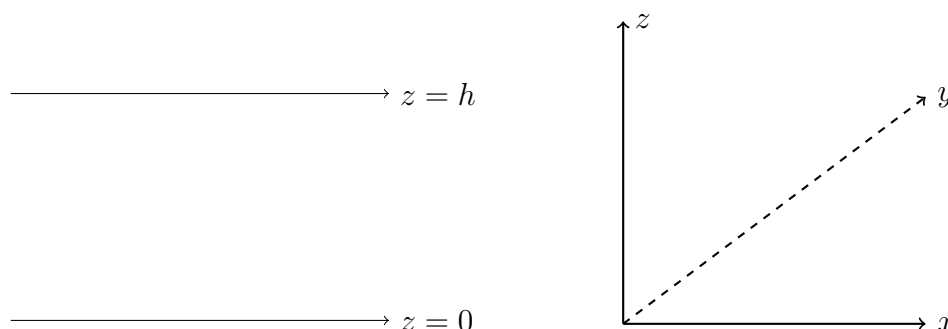
Figure 1: Geometry of the flow

Let the velocity of the flow is given by

$$\vec{q} = U(x.y,z)i, \ v = 0, \ w = 0. \tag{1}$$

From the continuity equation we have

$$\nabla \cdot \vec{q} = 0,$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{2}$$

since u is independent of x

$$\frac{\partial u}{\partial x} = 0. \tag{3}$$

The flow is extending to infinity in the y direction, the variation in this direction are neglected

$$\frac{\partial u}{\partial y} = 0, \tag{4}$$

therefore u is independent of y, hence

$$\vec{q} = u(z)i. \tag{5}$$

The Navier-Stokes equation [4, 5] is given by,

$$\rho \left[\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla)\vec{q} \right] = \rho\vec{g} - \nabla P + \mu_0(\vec{M} \cdot \nabla)\vec{H} + \mu\nabla^2\vec{q}. \tag{6}$$

For Irrotational flow we have

$$\nabla \times \vec{q} = 0, \ \vec{q} = -\nabla\phi. \tag{7}$$

Maxwell's equations [4, 5, 6] are given by

$$\nabla \cdot \vec{B} = 0, \ \vec{B} = \mu_0(\vec{H} + \vec{M}). \tag{8}$$

We have Magnetization[4, 5, 7] equation for ferrofluid as

$$\vec{M} = \overline{\mu}\vec{H}. \tag{9}$$

Since the flow is steady we have

$$\frac{\partial \vec{q}}{\partial t} = 0. \tag{10}$$

Consider

$$(\vec{q} \cdot \nabla)\vec{q} = \left(u(z)\frac{\partial}{\partial x}\right)u(z) = 0. \tag{11}$$

Since the body forces are absent we have

$$\vec{g} = 0. \tag{12}$$

Consider

$$\nabla^2\vec{q} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right),$$

using equations (2) and (4) we get

$$\nabla^2\vec{q} = \frac{\partial^2 u}{\partial z^2} = \frac{d^2 u}{dz^2}\hat{i}, \tag{13}$$

Let

$$\vec{M} = (M_x, M_y, M_z), \ \vec{H} = (H_x, H_y, H_Z), \tag{14}$$

consider

$$(\vec{M} \cdot \nabla)\vec{H} = \left[(M_x, M_y, M_z) \cdot \left(\frac{\partial}{\partial x}\hat{i}, \frac{\partial}{\partial x}\hat{j}, \frac{\partial}{\partial x}\hat{k}\right)\right](H_x, H_y, H_z),$$

$$(\vec{M} \cdot \nabla)\vec{H} = \left(M_x\frac{\partial H_x}{\partial x} + M_z\frac{\partial H_x}{\partial z}\right)\hat{i} + \left(M_x\frac{\partial H_x}{\partial x} + M_z\frac{\partial H_x}{\partial z}\right)\hat{k}, \tag{15}$$

substituting equation (15) in equation (6) we get

$$-\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial z^2}\right)\hat{i} + \mu_0\left(M_x\frac{\partial H_x}{\partial x} + M_z\frac{\partial H_x}{\partial z}\right)\hat{i} = 0, \tag{16}$$

$$-\frac{\partial p}{\partial x} + \mu_0\left(M_x\frac{\partial H_x}{\partial x} + \frac{\partial H_x}{\partial z}\right)\hat{k} = 0. \tag{17}$$

and from equations (7) and (8) we have,

$$H_x = -\frac{\partial \phi}{\partial x}, \quad H_y = -\frac{\partial \phi}{\partial z}, \quad H^2 = H_x^2 + H_y^2, \tag{18}$$

$$M_x = \overline{\mu}H_x \quad M_y = \overline{\mu}H_x \quad M_z = \overline{\mu}H_z. \tag{19}$$

Consider

$$M_x \frac{\partial H_x}{\partial x} + M_z \frac{\partial H_x}{\partial z} = \overline{\mu} \left(\frac{\partial \phi}{\partial x} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial \phi}{\partial z} \frac{\partial^2 \phi}{\partial x \partial z} \right),$$

$$= \frac{\overline{\mu}}{2} \frac{\partial}{\partial x} \left[\left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial z} \right)^2 \right],$$

$$= \frac{\overline{\mu}}{2} \frac{\partial}{\partial x} (H_x^2 + H_y^2),$$

$$M_x \frac{\partial H_x}{\partial x} + M_z \frac{\partial H_x}{\partial z} = \frac{\overline{\mu}}{2} \frac{\partial H^2}{\partial x}. \tag{20}$$

Now consider

$$M_x \frac{\partial H_z}{\partial x} + M_z \frac{\partial H_z}{\partial z} = \overline{\mu} \left(\frac{\partial \phi}{\partial x} \frac{\partial^2 \phi}{\partial x \partial z} + \frac{\partial \phi}{\partial z} \frac{\partial^2 \phi}{\partial x^2} \right),$$

$$= \frac{\overline{\mu}}{2} \frac{\partial}{\partial z} \left[\left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial z} \right)^2 \right],$$

$$= \frac{\overline{\mu}}{2} \frac{\partial}{\partial z} (H_x^2 + H_y^2)$$

$$M_x \frac{\partial H_z}{\partial x} + M_z \frac{\partial H_z}{\partial z} = \frac{\overline{\mu}}{2} \frac{\partial H^2}{\partial z}. \tag{21}$$

Nomenclature: $\vec{q}$ is the velocity of the fluid, ρ is the density of the fluid, $\vec{g}$ is the gravitation, P is the pressure, $\vec{M}$ is the magnetization, $\vec{H}$ is the magnetic field intensity, μ_0 is the permeability of free space, $\overline{\mu}$ is the magnetic susceptibility, η is the coefficient of viscosity of the fluid.

3   Method of Solution

Substituting equations (20) and (21) in equations (16) and (17) we get

$$\mu \frac{\partial^2 u}{\partial z^2} - \frac{\partial}{\partial x} \left(p - \frac{\overline{\mu} \mu_0 H^2}{2} \right) = 0. \tag{22}$$

$$\frac{\partial}{\partial z} \left(p - \frac{\overline{\mu} \mu_0 H^2}{2} \right) = 0. \tag{23}$$

Equation (22) can be written as

$$\mu \frac{\partial^2 u}{\partial z^2} - A = 0, \tag{24}$$

where

$$A = \frac{\partial}{\partial x} \left(p - \frac{\overline{\mu} \mu_0 H^2}{2} \right)$$

Integrating the equation (24) we get

$$\mu \frac{\partial u}{\partial z} - Az = C,$$

$$u = \frac{1}{\mu} \left[\frac{Az^2}{2} + Cz + D \right], \tag{25}$$

where C and D are constants to be determined.

Boundary Conditions:

Since the fluid is viscous and the upper rigid plate is moving with constant speed U while the lower rigid plate is stationary.

The no slip condition suggests that

$$u = 0 \text{ at } z = 0, \tag{26}$$

$$u = U \text{ at } z = h, \tag{27}$$

using equations (26) and (27) in equation (25) we get

$$\begin{aligned} D &= 0, \\ C &= \frac{U}{h} - \left(\frac{Ah}{2\mu}\right), \end{aligned} \tag{28}$$

therefore equation (25) becomes

$$\begin{aligned} \frac{u}{U} &= \frac{z}{h} - \frac{Ah^2}{2\mu U}\left(\frac{z}{h} - \frac{z^2}{h^2}\right), \\ u^* &= z^* - \frac{Ah^2}{2\mu U}(z^* - z^{*2}), \end{aligned} \tag{29}$$

where $u^* = \dfrac{u}{U}$ is the non dimensional velocity and

$z^* = \dfrac{z}{h}$ is the non dimensional vertical length.

Neglecting the asterisks (*) for simplicity we get

$$u = z + Gz(1 - z), \tag{30}$$

where $G = -\dfrac{Ah^2}{2\mu U}$. Hence equation (30) is the velocity distribution for Generalized plane Couette flow.

3.1   Maximum Velocity

Differentiating equation (30) w.r.t z and equating to zero we get

$$z = \frac{G+1}{2G}, \tag{31}$$

also,

$$\frac{\partial^2 u}{\partial z^2} = -2G \leq 0, \tag{32}$$

where $G \geq 0$. The maximum velocity is given by,

$$\begin{aligned} U_{max} &= \frac{G+1}{2G} + G\left(\frac{G+1}{2G}\right)\left(1 - \frac{G+1}{2G}\right), \\ U_{max} &= \frac{(G+1)^2}{4G}, \end{aligned} \tag{33}$$

3.2 Average Velocity

The average velocity is denoted by $\overline{U}$ is defined as the ratio of the total flow over a cross section to the area of the cross section and it is given by

$$\overline{u} = \frac{1}{h} \int_0^h u\,dz,$$
$$\overline{u} = \frac{1}{h} \int_0^h u^* U h\,dz^*, \qquad (34)$$
$$\overline{u}^* = \int_0^h u^*\,dz^*,$$

where $\overline{u}^* = \dfrac{\overline{u}}{U}$, is the non dimensional average velocity.
Neglecting the asterisks (*) for simplicity and simplifying we get

$$\overline{u} = \int_0^1 u\,dz,$$
$$= \int_0^1 [z + Gz(1-z)]\,dz, \qquad (35)$$
$$\overline{u} = \frac{\left(\left\{-\dfrac{\partial}{\partial x}\left(p - \dfrac{\mu_0 \overline{\mu} H^2}{2}\right)\dfrac{h^2}{2\mu U}\right\} + 3\right)}{6}.$$

3.3 Mass flow rate

The mass flow rate denoted by M, is defined as the amount of fluid that passes through any cross section of the channel per unit width and per unit time, it is given by

$$M = \rho \int_0^h u\,dz,$$
$$M = \rho h \frac{\left(\left\{-\dfrac{\partial}{\partial x}\left(p - \dfrac{\mu_0 \overline{\mu} H^2}{2}\right)\dfrac{h^2}{2\mu U}\right\} + 3\right)}{6}. \qquad (36)$$

4 Results and Discussions

Velocity distribution for generalized plane Couette flow , Average velocity and mass flow rate are determined for ferrofluid in the absence of pressure gradient. In the figure 2 we observe that for a decreasing pressure in the direction of motion the velocity is positive over the entire width between the plates, for an increasing pressure in the direction of motion shows that the velocity over a portion of the channel width can become negative, that is back flow may occur near the wall which is at rest. We found that the average velocity is constant, whereas mass flow rate is proportional to the density of the fluid.

5   Graphs
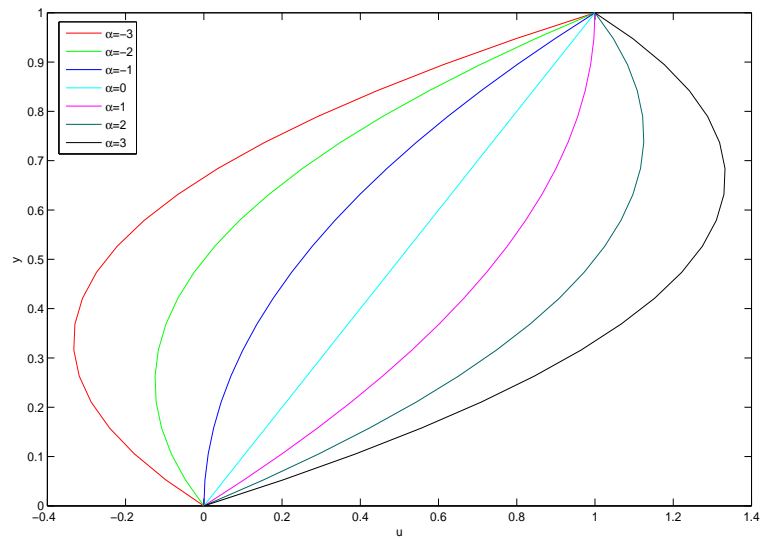
Figure 2: Velocity profile $\dfrac{u}{U}$ versus $\dfrac{z}{h}$

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Study on Ethical Hacking

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Abstract: *Nowadays most of the information of a person can be seen through the social media, which can be seen by everyone where there will be a chance of being hacked It is important for us to know about the hacking in today's IT world as everything is moving forward. Many people usually have a bad opinion about a hacker because of some hackers who have involved in some malicious activities. That is why many people don't have good opinion about the hacker. A study shows that almost 90% attacks have happen on the sides which shows that easy it is to invade into system or network for insiders. Here is when Ethical hacking comes into light. Many organizations hire ethical hackers to keep a track on their system and computer network [1].*

Hacking is an activity in which a person exploits the weakness in a system for self-profit or gratification. There is a need for protecting our system from hackers [2]. Ethical hacking is the way to find out the weakness and vulnerabilities in the system or computer network. Ethical hacker has a good purpose to do it. Though the ethical hacking has become a very upcoming technological subject from last few years, now the doubt remains the true intentions of the hacker [3], which will be discussed in this paper.

In this paper we will discuss about hacking, techniques of hacking, types of hacking, ethical hacking, phases of ethical hacking, types of ethical hacking, importance of use of ethical hacking, softwares used to hack the system. We have discussed about how our systems are being hacked and how to recover our system from being hacked. As mostly observed, usually social media accounts and bank accounts is being hacked everywhere. It is not just that we want everyone to know about our lives but is important to maintain security. There is a need of ethical hackers everywhere in this cyber crime world.

Keywords: *Vulnerabilities, Hacking, Ethical hacking, Cyber crime.*

1 Introduction

Hacking has been a part of computing for almost 5 decades and it is broad discipline which covers a wide range of topics. The first known event of hacking had taken place in 1960 at MIT and at the same time, the term **"hacker"** was originated.

Hacking is usually done to gain unauthorized access to a computer system or a computer network either to harm the system or to steal the sensitive information available on the computer. Hacking is usually legal as long as it is being done to find weakness in a computer or network system for testing purpose. This sort of hacking is what we call **"ETHICAL HACKING"**.

1.1 Types of Hacking

In the present world we can see many type of hacking activities that is going on, and there are many types of hackers. We can separate types of hacking into various categories based on what

is being hacked. Some of the types are as follows:

1. **Web Application Hacking:** Web application provides an interface between end users and web servers through a set of web pages generated at the server end. Hacking a web app means taking control over the server and its associated software such as database and other interfaces.

   Methods that can be used to hack web applications are SQL injection attacks, cross site scripting, cross site request forgeries, insecure communications etc. As an expert ethical hacker you need to test web applications for cross site scripting vulnerabilities, cookie hijacking, command injection attacks and secure web application from such attacks. [4]

2. **Network Hacking:** Attacking a network means collecting info about the network by using tools like nmap, NS lookup, ping, tracert etc. with intention to hack full network with the connected devices. These specially designed software programs generally manipulate data passing through a network connection in ways designed to obtain more information about how the target system works.

3. **Email Hacking:** It involves taking unauthorized access on an email account and using it without getting the consent of the user. There are number of ways in which a hacker can illegally gain access to an email account, some of them are virus and phishing [5].

4. **Password Hacking:** Password hacker refers to the individual who attempts to crack the secret word, phrase or string of characters used to gain access to secured data. Password hacking is often referred to as password cracking. In genuine case, the password hacker tries to recover password from data transmitted by or stored on a computer [6].

5. **Computer Hacking:** Basically it's a process of stealing machine id and password by applying hacking techniques and getting unauthorized access to a computer system. Hackers transform computers into zombies by using small programs that exploit the weakness in a computers operating system. In order to infect a computer a hacker must get the installation program to the victim [7].

1.2   What is Ethical Hacking?

An Ethical hacking includes discovering weakness in a machine or network system for testing purpose and lastly fixing them. A person involved in these activities in a legal way is called an ethical hacker. An ethical hacker also referred as white hat hacker, is an information security expert who systematically attempts to penetrate a computer system, network, application or other computery resource on behalf of its owners, and with their permission to find security vulnerabilities.

1.3   Techniques of Hacking

1. **Bait and Switch:** Using this hacking technique an attacker can buy advertising spaces on website. Later when a user clicks on the add ,he might get directed to a page which is infected with malware. This way, they can further install malware or adware on your computer. The hacker can run a malicious program which the user believes to be authentic. This way, after installing the malicious program on your computer, the hacker gets unprivileged access to your computer.

2. **Cookie Theft:** The cookies of a browser keep our personal data such as browsing history, user name and passwords for different sites that we access. Once the hacker gets the access to your cookie, he can even authenticate himself as you on a browser. A popular method to carry out this attack is to encourage a user's IP packets to pass through attackers machine.

3. **Clickjacking Attacks:** Clickjacking is also known by a different name ,UI redress.in this attack, the hacker hides the actual UI where the victim Is supposed to click. This behavior is very common in app download, movie streaming, and torrent websites. While they mostly employee this technique to earn advertising dollars, others use to steal the personal information.

4. **Virus, Trojan etc.:** Virus and trojan are malicious software programs which get installed into the victim's system and keeps sending the victims data to the hacker. They can also lock your files , serve fraud advertisement, divert the traffic , sniff your data, or spread on all the computer connected to your network.

5. **Phishing:** Phishing is a hacking technique using which a hacker replicated the most accessed sites and traps the victim by sending that spoofed link. Combined in social engineering, it becomes one of the most commonly used and deadliest attack vectors. Once the victim tries to login or enters some data, the hacker gets the private information of the target victim using the trojan running on the fake site.

6. **Eavesdropping (Passive attacks):** Unlike other attacks which are active in nature, using the passive attacker, a hacker just monitors the computer systems and networks to gain some unwanted information. The motive behind eavesdropping is not to harm system but to get some information without being identified.

7. **Fake Wap:** Even just for fun, a hacker can use software to fake a wireless access point(WAP). This WAP connects to the official public place WAP. Once you get connected to the fake WAP, a hacker can access your data. Its once of the easier hacks to accomplish and one just needs a simple software and wireless network.

8. **Waterhole Attacks:** Once hackers are aware of your timings, using this type of hacking, they might create a fake wi-fi access point and modify your most visited website to redirect them to you to get your personal information. As this attack collects information on a user from a specific place, detecting the attacker is even harder.

9. **Denial of Service (DoS/DDoS):** Denial of service attack is a hacking technique to take down a site or server by flooding that site or server with lot of traffic that the server is unable to process all the requests in the real time and finally crashes down. This popular technique, attacker floods the targeted machine with tons of request to overwhelm the resources, which, in turn, restrict the actual request from being fulfilled.

10. **Keyloggers:** Keylogger is a simple software that records the simple key sequence and strokes of your keyboard in a log file on your machine. These log files might even contain your personal email ids and passwords. Also known as keyboard capturing can be either software or hardware. While software based keyloggers target the programs installed on a computer, hardware devices targets keyboard, electromagnetic emissions, smart phone censors etc.

Keyloggers is one of the main reasons why online banking sites give you an option to use their virtual keyboards. [8]

11. **Spoofing Attack:** It involves websites which falsify data by mimicking legitimate sites(trusted) and they are therefore treated as trusted sites by users or other programs.

12. **Packet Sniffer:** Applications that capture data packets in order to view data and passwords in transit over networks.

1.4   Phases of Ethical Hacking

An ethical hacker follow these phases to hack into a system. It helps hackers to make a structured ethical hacking attack. The entire process can be categorized into following six phases.

1. **Reconnaissance:** It is also known as foot printing and info gathering phase, where the hacker collects as much as possible information about the target. The information here is collected in about three groups: network, host and the people involved.
   There are two types of foot printing:

   ACTIVE: Directly interacting with the target to gather info about the target.

   PASSIVE: Collect information without directly accessing the target and it involves collecting information from social media etc.

2. **Scanning:** There are 3 types of scanning namely

   a) Port Scanning: It involves scanning the target for the information like open ports, live systems, various services running on the host.

   b) Vulnerability Scanning: Checking the target for weakness or vulnerabilities which can be exploited.

   c) Network Mapping: Finding topology of a network, routers, firewalls, serves if any and host information and drawing a network diagram with the available information.

3. **Gaining Access:** This phase is where an attacker breaks into the system or network using various tools or methods. After entering into a system, he has to increase his privilege to administrator level so he can install an application he needs or modify or hide data.

4. **Maintaining Access:** Maintaining access to the target until he finishes the task he planned to accomplish in the target. After gaining access, the hacker installs some backdoors in order to enter into the system when he needs access in this owned system in future. Metasploit is one of the tool used in this process.

5. **Clearing Track:** An intelligent hacker always clears all evidences so that in the later point of time, no one will find any traces leading to hacker. This process is actually an unethical activity. It has to do with the deletion of logs of all the activities that takes place during the hacking process.

6. **Reporting:** It is the last step of finishing the ethical hacking process. Here the ethical hacker compiles a report with his findings and the job that was done such as the tools used, the success rate, vulnerabilities found and the exploit processes.

1.5   Types of Ethical Hacking

Hackers can be classified into different categories such as white hat, black hat and grey hat, based on their intent of hacking a system.

i) **White hat hackers:** White hat hackers are also known as ethical hackers. They never intent to harm a system, rather they try to find out weakness in a computer or network system as apart of penetration testing and vulnerabilities assessments.

ii) **Black hat hackers:** Black hat are also known as crackers ,are those who hacks in order to gain unauthorized access to a system and harm its cooperation and steal sensitive information. It is always illegal because of its bad intent which includes stealing corporate data, violating privacy, damaging the system, blocking network communication, etc.

iii) **Grey hat hackers:** Grey hat hackers are a blend of both black hat and white hat hackers. They act without malicious intent but for their fun, they exploit a security weakness in a computer system or network without the owner's permission or knowledge. Their intent is to bring the weakness to the attention of the owners and getting appreciation or a little bounty from the owners.

iv) **Red hat hackers:** Red hat hackers are again a blend of both black hat and white hat hackers. They usually on the level of hacking government agencies, top agencies, top secret information hubs, and generally anything that falls under the category of sensitive information.

v) **Blue hat hackers:** A blue hat hacker is someone outside computer security consulting firms who is used to bug -test a system proper to its launch. They look for loopholes that can be exploited and try to close these gaps.

vi) **Elite hackers:** This is a social status among hackers, which is used to describe the most skilled or newly discovered exploits will circulate among these hackers.

vii) **Script Kiddie:** It is a non-expert who breaks into a computer systems by using pre-packaged automated tools written by others , usually with the little understanding of the underlying concept , hence the term kiddie.

viii) **Neophyte:** A neophyte, n00b, newbie or green hat hacker is someone who is new to hacking or phreaking and has almost no knowledge or experience of the workings of technology and hacking.

ix) **Hacktivist:** A hacktivist is a hacker who utilizes technology to announce a social, ideological, religious or political message. In general, most hacktivism involves website defacement or denial-service attacks [9].

1.6   Why Ethical Hacking is Necessary?

In today's digital era, one of the biggest threats comes from cyber criminals. Hackers were not taken seriously until a few years back . Recently, some big names in the Indian industry had to pay hefty sums of money to hackers to keep confidential information from being disclosed to the government. According to surveys conducted by cyber security firms in the country, Indian firms lost more than 4 billion in 2013 alone because of hackers. Cyber crimes are becoming more common in todays technology world. The need of ethical hackers are mainly in MNC, huge organizations where their large and confidential data must be protected from being hacked or leaked. Ethical hacking as a career offers immense opportunities. [10]

1.7   How to know that our system is begin hacked?

Hacking has become very common now a days. Everyone is interested to know other person details or to earn money illegally. Some of them are unable to detect that their system is being hacked. Here we learn how to know that our system is being hacked.

1. **Ransom Message:** It is a malware, that locks up your precious data and asks for a payment using online currency to get it back. If you're lucky, that ransomware message might go away with a quick reboot. Some of the ransomware messages come from programs that don't lock up your data.

2. **Fake Antivirus Messages:** In moderate decline these days, fake antivirus warning messages are among the surest signs that your system is being compromised. What most people don't realize is that by the tome they see the fake antivirus warning, the damage has been done. Clicking no or cancel to stop the fake virus scan is too little, too late. The malicious software has already made use of unpatched software, often a browser add-on program, to completely exploit your system.

3. **Unwanted Browser Toolbars:** This is a very common sign of exploitation. Your browser has multiple new toolbars with names that seem to indicate the toolbar is supposed to help you. Unless you recognize the toolbar is coming from a very well known vendor, its time to demo the bogus toolbar.

4. **Redirected Internet Searches:** Many hackers make their living by redirecting your browser somewhere other than you want to go. The hacker gets paid by getting your clicks to appear on someone else's websites, often those who don't know that the clicks to their site are from malicious redirection.

5. **Frequent Random Popups:** This is a popular sign that you've been hacked, is also one of the more annoying ones. When you're getting random browser pop-ups from websites that don't normally generate the, your system has been compromised.

6. **Your friends receive social media invitations from you that you didn't send:** Either you or your friends receive invitations to be a friend when you are already connected friends on that social media site. Then you notice the new friend's social media site is devoid of other recognizable friends and none of the older posts. In either case, the hacker either controls our social media site, has created a second near look alike bogus page or you or friend has installed a rouge social media application.

7. **Your online isn't working:** If you are typing in your online password correctly, for sure, and it isn't working, then you might be hacked. Then a hacker has logged in using it and changed it to keep you out.

8. **Unexpected Software Installs:** Unwanted and unexpected software installs are a big sign that your computer system has likely been hacked. For whatever reason, most malware programs these days are trojans and worms and they typically install themselves like legitimate programs.

9. **Your mouse moves between programs and makes correct solutions:** If your mouse pointer moves itself while making selections that work, you've definitely been hacked. Mouse pointers often move randomly usually due to hardware problems. If the movements involve making the correct choices to run particular programs, malicious humans are somewhere involved.

10. **Your anitmalware software, task manager or registry editor is disabled and cannot be restarted:** This is a huge sign of malicious compromise. If you notice that your antivirus software is disabled and you didn't do it, you're probably exploited, especially if you try to start task manager or registry editor and they wont start, or start and disappear, or start in a reduced state. This is very common for malware to do. [11]

1.8   How to recover our system from hacking

- Isolate your system.
- Shutdown and remove the hardware.
- Scan your drive for infection and malware.
- Backup your important files from the previously infected drive.
- Move your drive back to your PC.
- Completely wipe your old hard drive.
- Reload the operating system from trusted media and install update.
- Re-install antivirus, antispyware and other security software.
- Scan your data backup disks for viruses.
- Make a complete backup of your system. [12]

1.9   Disadvantages of Ethical Hacking

- The ethical hacker using the knowledge they gain to do malicious hacking activities.
- Allowing the company's financial and banking details to be seen.
- The possibility that the ethical hacker will send and/or place malicious code, viruses, malware and other destructive and harmful things on a computer system.
- Massive security breach. [13]

2   Conclusion

Hacking is not a crime! It depends on the users mindset. This paper helps one to know about how to detect whether the system is being hacked or not and try to recover the system . It is just information given how system being hacked, as we all know this basic knowledge to keep our system safe in this present technical world.

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Study of Mellin Transforms and its Applications

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Abstract: We deal with the definition and examples of Mellin transforms, the basic operational properties of Mellin transforms. We have given an insight to the applications of Mellin transform and discussed the same using examples. We have also applied Mellin transform to summation of infinite series.

Keywords: Integral transformations, Mellin transforms, Inverse Mellin transforms, Gamma functions, Zeta functions, Properties of Mellin transforms, Applications to infinite series.

AMS Subject Classification: 44A15.

1   Introduction

In integral Transform, we transform the unknown function, $f(x)$, to a different function say $F(t)$. Two of the most famous and widely used integral transforms are the Fourier transforms and the Laplace transforms [1, 2]. The Mellin transform is an integral transform that is regarded as the multiplicative version of the two-sided Laplace transform. This integral transform is closely connected to the theory of Dirichlet series, and is often used in number theory, mathematical statistics, and the theory of asymptotic expansions. The Mellin Transform and its inverse are derived from the complex Fourier transform and its inverse respectively.

Definition of Mellin transform and inverse Mellin transform

The Mellin transform of the function $f(x)$ is defined as follows [3, 4] :

$$M\{f(x)\} = \tilde{f}(p) = \int_0^\infty x^{p-1} f(x) dx \tag{1}$$

The inverse Mellin transform of the function $f(x)$ is defined as :

$$M^{-1}\{\tilde{f}(p)\} = f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{p-1} f(p) dp \tag{2}$$

where $f(x)$ is a real valued function defined on $(0, \infty)$ and the Mellin transform variable p is a complex number. M and M^{-1} are linear integral operators.

Mellin transform is closely related to the Laplace transform and the Fourier transform, and the theory of the gamma function and allied special functions. The gamma function is commonly used extension of the factorial function to complex numbers which is given by [5]

$$\Gamma(p) = \int_0^\infty e^{-t} t^{p-1} dt \tag{3}$$

2   Properties of Mellin transforms

Theorem 1 (Scaling Property). *If $M\{f(x)\} = \tilde{f}(p)$, then $M\{f(ax)\} = a^{-p}\tilde{f}(p)$, $a \geq 0$.*

Proof. By the definition of Mellin transform and substituting $ax = t$, we have

$$M\{f(ax)\} = \frac{1}{a^p} \int_0^\infty t^{p-1} f(t) dt$$

$$= \frac{\tilde{f}(p)}{a^p}$$

□

Theorem 2 (Mellin Transforms of Derivatives). *If $M\{F(x)\} = \tilde{f}(p)$, then*
- $M\{f'(x)\} = -(p-1)\tilde{f}(p-1)$ *provided $x^{p-1}f(x)$ vanishes as $x \to 0$ and as $x \to \infty$.*
- $M\{f''(x)\} = (p-1)(p-2)\tilde{f}(p-2)$.
- *More generally ,$M\{f^{(n)}(x)\} = (-1)^n \dfrac{\Gamma(p)}{\Gamma(p-n)}\tilde{f}(p-n)$ provided $x^{p-r-1}f^{(r)}(x) = 0$ as $x \to 0$ for $r = 0, 1, 2, \ldots, (n-1)$ [6, 7].*

Proof. By the definition of Mellin transform and integrating the integral by parts, we obtain

$$M\{f'(x)\} = \left[x^{p-1}f(x)\right]_0^\infty - (p-1)\int_0^\infty x^{p-2}f(x)dx$$

$$= (-1)^1 (p-1)\tilde{f}(p-1)$$

Again consider,

$$M\{f''(x)\} = \int_0^\infty x^{p-1}f''(x)dx$$

$$= (-1)^2 (p-1)(p-2)\tilde{f}(p-2)$$

$$= (-1)^2 \frac{\Gamma(p)}{\Gamma(p-2)}\tilde{f}(p-2) \qquad \left(\text{Since } \Gamma(p+1) = p\Gamma(p)\right)$$

Proceeding this way to n times we get, $M\{f^{(n)}(x)\} = (-1)^n \dfrac{\Gamma(p)}{\Gamma(p-n)}\tilde{f}(p-n)$.   □

Theorem 3. *If $M\{F(x)\} = \tilde{f}(p)$, then*
- $M\{xf'(x)\} = -\tilde{f}(p)$, *provided $x^p f(x)$ vanishes as $x = 0$ and as $x \to \infty$.*
- $M\{x^2 f''(x)\} = (-1)^2 p(p+1)\tilde{f}(p)$.
- *More generally, $M\{x^n f^{(n)}(x)\} = (-1)^n \dfrac{\Gamma(p+n)}{\Gamma(p)}\tilde{f}(p)$.*

Proof. By the definition of Mellin transform and integrating the integral by parts, we obtain

$$M\{f'(x)\} = \left[x^p f(x)\right]_0^\infty - p\int_0^\infty x^{p-1}f(x)dx$$

$$= -p\tilde{f}(p).$$

Again consider,

$$M\{x^2 f''(x)\} = \int_0^\infty x^{p-1} x^2 f''(x) dx$$

$$= (-1)^2 p(p+1)\tilde{f}(p).$$

Proceeding this way to n times we get, $M\{x^n f^{(n)}(x)\} = (-1)^n \dfrac{\Gamma(p+n)}{\Gamma(p)}\tilde{f}(p).$ $\qquad \square$

Theorem 4 (Mellin Transforms of Differential Operators). *If $M\{F(x)\} = \tilde{f}(p)$, then*

- $M\left\{\left(x\dfrac{d}{dx}\right)^2 f(x)\right\} = (-1)^2 p^2 \tilde{f}(p)$

- *More generally, $M\left\{\left(x\dfrac{d}{dx}\right)^n f(x)\right\} = (-1)^n p^n \tilde{f}(p).$*

Proof. Consider

$$M\left\{\left(x\dfrac{d}{dx}\right)^2 f(x)\right\} = M\{x^2 f''(x) + x f'(x)\}$$

$$= p(p+1)\tilde{f}(p) - p\tilde{f}(p) \qquad \left(\text{By Theorem } 3\right)$$

$$= (-1)^2 p^2 \tilde{f}(p).$$

Proceeding in the same way for n times we get, $M\left\{\left(x\dfrac{d}{dx}\right)^n f(x)\right\} = (-1)^n p^n \tilde{f}(p).$ $\qquad \square$

Theorem 5 (Convolution Theorem). *If $M\{f(x)\} = \tilde{f}(p)$ and $M\{g(x)\} = \tilde{g}(p)$, then $M\{f(x) * g(x)\} = \tilde{f}(p)\,\tilde{g}(p)$.*

Proof. Using the convolution property and the definition of Mellin transform, we have

$$M\{f(x) * g(x)\} = M\left\{\int_0^\infty f(\xi) g\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi}\right\}$$

$$= \int_0^\infty f(\xi)\frac{d\xi}{\xi} \int_0^\infty x^{p-1} g\left(\frac{x}{\xi}\right) dx$$

$$= \int_0^\infty \xi^{p-1} f(\xi) d\xi \int_0^\infty (u)^{p-1} g(u) du \qquad \left(\text{Substituting } \frac{x}{\xi} = u\right)$$

$$= \tilde{f}(p)\,\tilde{g}(p)$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Theorem 6 (Parseval's Type Property). *If $M\{f(x)\} = \tilde{f}(p)$ and $M\{g(x)\} = \tilde{g}(p)$, then*

$$M\{f(x)g(x)\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(s)\tilde{g}(p-s) ds.$$

Proof. By definition of Mellin transform, we have

$$M\{f(x)g(x)\} = \int_0^\infty x^{p-1} f(x)g(x)dx$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{f}(s)\tilde{g}(p-s)ds \quad \left(\text{Using equation(2)}\right)$$

When $p = 1$, the above result reduces to $\int_0^\infty f(x)g(x)dx = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{f}(s)\tilde{g}(1-s)ds$, which is the Parseval formula for the Mellin transform. $\square$

3 Examples on Mellin transforms

Example 1: Find the Mellin transform of the folowing functions:

(i) $f(x) = e^{-nx}$
(ii) $f(x) = \dfrac{2}{e^{2x} - 1}$
(iii) $f(x) = \dfrac{1}{(1+x)^n}$

(i) Using the definition of Mellin Transform and substituting $nx = t$, we get

$$M\{e^{-nx}\} = \frac{1}{n^p} \int_0^\infty t^{p-1} e^{-t}\, dt \tag{4}$$

$$= \frac{\Gamma(p)}{n^p} \quad \left(\text{from equation (3)}\right)$$

Thus $\dfrac{\Gamma(p)}{(ik)^p} = \dfrac{\Gamma(p)}{k^p} \left[\cos\dfrac{p\pi}{2} - \sin\dfrac{p\pi}{2}\right]$. Separating the real and imaginary part we get,

$$M\{\cos kx\} = k^{-p}\Gamma(p)\cos\left(\frac{p\pi}{2}\right) \text{ and } M\{\sin kx\} = k^{-p}\Gamma(p)\sin\left(\frac{p\pi}{2}\right)$$

.

(ii) Using the definition of Mellin Transform we have

$$M\left\{\frac{2}{e^{2x} - 1}\right\} = 2\sum_{n=1}^\infty \int_0^\infty x^{p-1} e^{-2nx} dx \quad \left(\text{Since } \sum_{n=1}^\infty e^{-nx} = \frac{1}{e^{-x} - 1}\right)$$

$$= 2^{1-p}\Gamma(p)\zeta(p). \tag{5}$$

where $\zeta(p) = \sum_{n=1}^\infty \dfrac{1}{n^p}, \quad Re(p) > 1$ is the Riemann Zeta function.

(iii) Using the definition of Mellin Transform and substituting $x = \dfrac{t}{1-t}$, we obtain

$$M\left\{\frac{1}{(1+x)^n}\right\} = \int_0^1 t^{p-1}(1-t)^{n-p-1}dt$$

$$= B(p, n - p)$$

$$= \frac{\Gamma(p)\Gamma(n - p)}{\Gamma(n)}$$

where $B(p, q)$ is the standard Beta function.

4   Applications of Mellin transforms

The Mellin transform has special importance in scale representation of signal because it is a scale invariant transform. The translation invariance property of the Fourier transform along with the scale-invariance property of the Mellin transform, provides a way of representing signal free of Doppler distortion. The Fourier-Finite Mellin transform may be applied in image processing, pattern recognition, speech processing, radar signal analysis etc. It is useful for resolution of certain types of classical boundary and initial value problems. The Mellin transform is an essential tool in studying the distributions of products of random variables, in probability theory. Inverse Mellin transforms commonly occur in Riesz means. The Mellin transform can be used in Audio timescale-pitch modification and also useful for summation of infinite series [7, 8].

Application of Mellin transforms to summation of infinite series

In this section we explore some uses of Mellin transforms in obtaining analytic and asymptotic information about infinite sums and integrals involving a parameter. The Riemann Zeta function plays a central role in this.

Example 2: Prove the relation $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = (1 - 2^{1-p})\zeta(p)$.

Hence deduce $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4} = \frac{7\,\pi^4}{720}$.

Solution: Consider $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ and multiply it by t^n, we obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}t^n}{n^p} = \frac{1}{\Gamma(p)} \int_0^{\infty} x^{p-1}dx \sum_{n-1}^{\infty}(-1)^{n-1}t^n e^{-nx} \qquad \left(\text{By equation (4)} \right)$$

$$= \frac{1}{\Gamma(p)} \int_0^{\infty} x^{p-1} \frac{t}{e^x + t}dx$$

Applying the limit as $t \to 1$, the above result gives

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = \frac{1}{\Gamma(p)} M\left\{ \frac{1}{e^x + 1} \right\}$$

$$= (1 - 2^{1-p})\zeta(p) \qquad \left(\text{Since } M\left\{ \frac{1}{e^x + 1} \right\} = (1 - 2^{1-p})\,\Gamma(p)\,\zeta(p) \right)$$

(i) Consider $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{1}{2}\zeta(2) = \frac{\pi^2}{12}$

(ii) Consider $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4} = \frac{7}{8}\zeta(4) = \frac{7\pi^4}{720}$

Example 3: Prove the following results :

(i) $\displaystyle M\left\{ \int_0^{\infty} \xi^n f(x\xi)g(\xi)d\xi \right\} = \tilde{f}(p)\,\tilde{g}(1+n-p)$

(ii) $\displaystyle M\left\{ \int_0^{\infty} \xi^n f\left(\frac{x}{\xi}\right)g(\xi)d\xi \right\} = \tilde{f}(p)\,\tilde{g}(p+n+1)$

By using the convolution theorem, we have

(i) $\displaystyle M\left\{ \int_0^{\infty} \xi^n f(x\xi)g(\xi)d\xi \right\} = \int_0^{\infty} \frac{\eta^{n-1}}{\xi^{n-1}}\frac{d\eta}{\xi} \int_0^{\infty} \xi^n f(\eta)g(\xi)d\xi \quad \left(\text{substituting } x\xi = \eta\right)$

$$= \tilde{f}(p)\,\tilde{g}(1+n-p)$$

(ii) $\displaystyle M\left[\int_0^{\infty} \xi^n f\left(\frac{x}{\xi}\right)g(\xi)d\xi \right] = \int_0^{\infty} \xi^{p-1}\eta^{p-1}\xi d\eta \int_0^{\infty} \xi^n f(\eta)g(\xi)d\xi \quad \left(\text{substituting } \frac{x}{\xi} = \eta\right)$

$$= \tilde{f}(p)\,\tilde{g}(p+n+1)$$

Example 4: Sum the series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos an}{n^2}$ and hence deduce $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Let $f(x) = \dfrac{\cos ax}{x^2}$, then $M\{f(x)\} = -\dfrac{\Gamma(s-2)}{a^{s-2}}\cos\dfrac{\pi s}{2}$. Thus we have,

$$\sum_{n=1}^{\infty} \frac{\cos an}{n^2} = -M^{-1}\left\{ \frac{\zeta(s)}{a^{s-2}}\Gamma(s-2)\cos\left(\frac{\pi s}{2}\right) \right\}$$

$$\sum_{n=1}^{\infty} \frac{\cos an}{n^2} = -\frac{a^2}{2}M^{-1}\left\{ \left(\frac{2\pi}{a}\right)^2 \frac{\zeta(1-s)\Gamma(s-2)}{\Gamma(s)} \right\}$$

There are three simple poles at $s = 0, 1, 2$, whose residues are given by

$$\text{Res}\,\{0\} = -\frac{1}{2},\ \text{Res}\,\{1\} = \frac{\pi}{a},\ \text{Res}\,\{2\} = -\frac{\pi^2}{3a^2}$$

Combining the results , we finally obtain the solution

$$\sum_{n=1}^{\infty} \frac{\cos an}{n^2} = -\frac{a^2}{2}\left(-\frac{1}{2} + \frac{\pi}{a} - \frac{\pi^2}{3a^2} \right)$$

$$= \frac{a^2}{4} - \frac{\pi a}{2} + \frac{\pi^2}{6}$$

Applying the limit as $a \to 0$ in the above expression we get, $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

Solution of Drichlet Boundary Value Problem by Mellin Transforms

Consider the transformed Laplace equation [9]

$$\frac{\partial^2 W}{\partial r^2} + \frac{1}{r} \frac{\partial W}{\partial r} + \frac{1}{r^2} \frac{\partial^2 W}{\partial \theta^2} = 0 \quad r \geq 0, \, 0 \leq \theta \leq \pi \tag{6}$$

with the conditions

$$w(r, \pi) = \begin{cases} T & \text{for } 0 \leq r \leq 1 \\ 0 & \text{for } r > 1 \end{cases} \qquad w(r, 0) = \begin{cases} 2T & \text{for } 0 \leq r \leq 1 \\ 0 & \text{for } r > 1 \end{cases}$$

The asymptotic behaviour of $w(r, 0) \approx r^0$ as $r \to 0$ and $w(r, \pi) \approx r^{\frac{-1}{2}}$ as $r \to \infty$.

Using the Mellin transform definition we have

$$\tilde{w}(s, \theta) = \int_0^{\infty} r^{s-1} w(s, \theta) dr$$

Equation (6) and the conditions are transformed into

$$\frac{d^2 \overline{w}}{d\theta^2} + s^2 \overline{w} = 0$$
$$\tilde{w}(s, \pi) = \frac{T}{s} \text{ and } \tilde{w}(s, 0) = \frac{2T}{s} \tag{7}$$

Equation (7) is an ordinary differential equation whose solution is given by

$$\overline{w}(s, \theta) = A \sin s\theta + B \cos s\theta \tag{8}$$

Solving for A and B, we get

$$\tilde{w}(s, \theta) = \frac{T(1 - 2 \cos \pi s)}{\sin \pi s} \frac{\sin s\theta}{s} + \frac{2T}{s} \cos s\theta \tag{9}$$

Solution to the original problem:

Consider the inverse formula in the form

$$\tilde{w}(r, \theta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{w}(s, \theta) r^{-s} ds, \, 0 < c < \frac{1}{2} \tag{10}$$

where $\tilde{w}(s, \theta)$ is given by (9). Using the Cauchy principal value, we evaluate the integral in (10) and the simple poles of $\tilde{w}(s, \theta)$ are located at $s = \pm n, n = 1, 2, 3, \ldots$.

At the poles $s = n$, the residues are given by

$$\tilde{w}(r, \theta)r^{-s} = \lim_{s \to n}(s - n)\tilde{w}(r, \theta)r^{-s}$$

$$= \frac{T}{\pi} \lim_{s \to n} \left[\frac{1}{\cos \pi s}\left(\frac{1 - 2\cos \pi s}{s}\right) \sin s\theta + 0\right] r^{-s} \quad \text{(Using (9))}$$

$$= \frac{T}{\pi}\left[(-1)^n - 2\right] \frac{\sin n\theta}{n} r^{-n}$$

For $n = 1, 2, 3, 4, \ldots$, we obtain $\tilde{w}(r, \theta) = \dfrac{T}{\pi} \displaystyle\sum_{n=1}^{\infty} \left[\dfrac{(-1)^n - 2}{n}\right] \sin n\theta r^{-n}$.

At the poles $s = -n$, the residues are given by

$$\tilde{w}(r, \theta)r^{-s} = \lim_{s \to -n}(s + n)\tilde{w}(s, \theta)r^{-s}$$

$$= \frac{T}{\pi} \lim_{s \to -n} \left[\frac{1}{\cos \pi s}\left(\frac{1 - 2\cos \pi s}{s}\right) \sin s\theta \, r^{-s}\right] \quad \text{(Using (9))}$$

$$= \frac{T}{\pi}(-1)^n \left[\frac{(-1)^n - 2}{n}\right] r^n \sin n\theta$$

For $-n = 1, 2, 3, 4, \ldots$, we obtain $\tilde{w}(r, \theta) = \dfrac{T}{\pi} \displaystyle\sum_{n=1}^{\infty} \left[\dfrac{(-1)^n - 2}{n}\right] \sin n\theta r^n$.

At $s = 0$, we have

$$\tilde{w}(r, \theta)r^{-s} = \lim_{s \to n}(s - 0)\tilde{w}(r, \theta)r^{-s}$$

$$= \lim_{s \to n} \left[\frac{T}{\pi} \frac{\pi s}{\sin \pi s}(1 - 2\cos \pi s)\theta \frac{\sin s\theta}{\theta s} + 2T \cos s\theta\right] \quad \text{(Using (9))}$$

$$= 2T - \frac{\theta T}{\pi}$$

Hence the solution is given by

$$w(r, \theta) = \begin{cases} 2T - \theta\dfrac{T}{\pi} + \dfrac{T}{\pi}\displaystyle\sum_{n=1}^{\infty}\left[\dfrac{(-1)^n - 2}{n}\right]r^n \sin n\theta & \text{for } r < 1 \\[4mm] \dfrac{T}{\pi}\displaystyle\sum_{n=1}^{\infty}\left[\dfrac{(-1)^n - 2}{n}\right]r^{-n}\sin n\theta & \text{for } r > 1 \end{cases} \quad (11)$$

To verify that equation (11) **is the solution of** (6)**.**

Case (i) For $r < 1$, equation (6) becomes

$$\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} = \frac{T}{\pi}\sum_{n=1}^{\infty}(n-1)[(-1)^n - 2]r^{n-2}\sin n\theta$$

$$+ \frac{T}{\pi}\sum_{n=1}^{\infty}[(-1)^n - 2]r^{n-2}\sin n\theta - \frac{T}{\pi}\sum_{n=1}^{\infty}n[(-1)^n - 2]r^{n-2}\sin n\theta$$

$$= 0$$

Also $w(r, \pi) = T$ and $w(r, 0) = 2T$. Thus the Laplace equation and the boundary conditions are satisfied for $r < 1$.

Case (ii) For $r > 1$, equation (6) becomes

$$\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} = \frac{T}{\pi}\sum_{n=1}^{\infty}(n-1)[(-1)^n - 2]r^{-n-2}\sin n\theta$$

$$-\sum_{n=1}^{\infty}[(-1)^n - 2]r^{-n-2}\sin n\theta - \sum_{n=1}^{\infty}n[(-1)^n - 2]r^{-n-2}\sin n\theta$$

$$= 0$$

Also $w(r, \pi) = 0$ and $w(r, 0) = 0$. Thus the Laplace equation and the boundary conditions are satisfied for $r > 1$.

5   Results and Discussions

We have studied about a type of integral transform called Mellin transform. Using the definition of Mellin transform and inverse Mellin transform we have proved a few properties and solved a few examples.We have also seen how to apply Mellin transforms to find the summation to infinite series and also obtain solution for Dirichlet Boundary Value Problem by Mellin Transforms.

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Magnetic fluid flow between two parallel plates in the absence of pressure gradient

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Abstract: The behaviour of ferrofluid in the absence of pressure gradient is analysed between parallel plates. Average velocity and mass flow rate are determined.

Keywords: Ferrofluids, Couette flow, maximum velocity, average velocity and mass flow rate.

AMS Subject Classification: 76DXX.

1   Introduction

A ferrofluid or magneticfluid is a liquid that become strongly magnetised in presence of a magnetic field. Ferrofluids are colloidal liquids made up of nano scale ferromagentic or ferrimagnetic particles suspended in a carrier fluid. It is an artificial material rather than formed naturally. Ferrofluid is superparamagnetic and create liquid seals held in position by magnetic fields [1, 2, 3].

Magnetic fluids are used to form liquid seals around the spinning drive shafts in hard disks, the rotating shaft is surrounded by magnets. A small amount of ferrofluid, placed in the gap between the magnet and the shaft, will be placed by its attraction of the magnet. The fluid of magnetic particles forms a barrier which prevents bits from entering the interior of the hard drive [1].

Magnetic fluids have friction-reducing capabilities. If applied to the surface of a strong enough magnet, such as one made of neodymium, it can cause the magnet to glide across smooth surfaces with minimal resistance [4]. It can also be used in semi-active dampers in mechanical and aerospace applications. While passive dampers are generally bulkier and designed for a particular vibration source in mind, active damper consume more power. Magnetic fluids based dampers solved both of these issues and are becoming popular in the helicopter community, which as to deal with large inertial and aerodynamic vibrations [5].

Magnetic fluids can be used to image magnetic domain structures on the surface of ferromagnetic materials using a technique developed by Francis Bitter [1]. Magnetic fluids are commonly used in loudspeaker to remove heat from the voice coil, and to passively damp the movement of the cone there reside in what would normally be the air gap around the voice coil, held in space by the speaker's magnet. Since the Magnetic fluid are paramagnetic,they obey Curie's law and thus become less magnetic at higher temperatures. A strong magnet placed near the voice coil will attract the cold ferrofluid more than the hot magnetic fluids thus forcing the heated ferrofluid away from the electric voice coil and towards a heat sink [1].

Agrawal [6] has studied the ferrofluid with an inclined lubricated porous slider bearing in the presence of an external applied magnetic field that is bent to the lower surface of the bearing.

They observed that the load capacity of the ferrofluid is greater than the viscous fluid and found that the ferrofluid is performance is superior to the viscous fluid. Raj et al. [7] showed that the ferrofluids are applied to the damping technology devices such as rotary viscous inertia damper, linear damper, damper assembly, and dashpot.

2   Geometry and Governing equations

Consider a steady flow of an incompressible viscous fluid between two infinite parallel rigid plates separated by a distance h. The upper plate moves with uniform velocity U_i while the lower plate is stationary with zero pressure gradient. Such a flow is called Plane Couette Flow [6, 8, 9, 10].

Let x be the direction of the flow. z be the direction perpendicular to the flow.
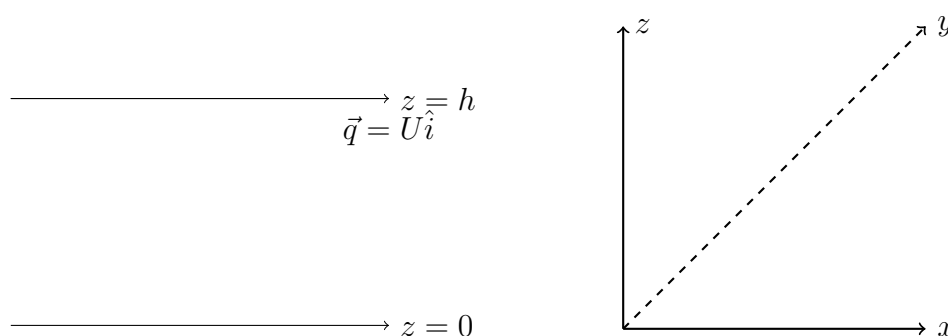
Figure 1: Geometry of the flow

The velocity of the flow is given by,

$$\vec{q} = U(x, y, z)\hat{i}, \ v = 0, \ w = 0, \tag{1}$$

From the continuity equation we have

$$\nabla \cdot \vec{q} = 0,$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{2}$$
$$\text{since } u \text{ is independent of } x, \frac{\partial u}{\partial x} = 0.$$

The flow is extending to infinity in the y direction, the variation in this direction are neglected and u is independent of y

$$\frac{\partial u}{\partial y} = 0. \tag{3}$$

Therefore we get

$$\vec{q} = u(z)\hat{i}. \tag{4}$$

The Navier-Stokes equation [4] is given by,

$$\rho \left[\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla)\vec{q} \right] = \rho\vec{g} - \nabla P + \mu_0(\vec{M} \cdot \nabla)\vec{H} + \eta\nabla^2\vec{q}. \tag{5}$$

Since the flow is steady, we have

$$\frac{\partial \vec{q}}{\partial t} = 0, \tag{6}$$

Consider,

$$(\vec{q} \cdot \nabla)\vec{q} = \left(u(z) \frac{\partial}{\partial x} \right) u(z)\hat{i} = 0, \tag{7}$$

Since the body force are absent, we have

$$\vec{g} = 0, \tag{8}$$

Consider,

$$\nabla^2 \vec{q} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \hat{i},$$

$$\nabla^2 \vec{q} = \frac{\partial^2 u}{\partial z^2}\hat{i} = \frac{d^2 u}{dz^2}\hat{i}, \tag{9}$$

Let

$$\vec{M} = (M_x, M_y, M_z), \quad \vec{H} = (H_x, H_y, H_z). \tag{10}$$

Consider

$$(\vec{M} \cdot \nabla)\vec{H} = \left[(M_x, M_y, M_z) \cdot \left(\frac{\partial}{\partial x}\hat{i}, \frac{\partial}{\partial y}\hat{j}, \frac{\partial}{\partial z}\hat{k} \right) \right] (H_x, H_y, H_z),$$
$$(\vec{M} \cdot \nabla)\vec{H} = \left(M_x \frac{\partial H_x}{\partial x} + M_z \frac{\partial H_x}{\partial z} \right) \hat{i} + \left(M_x \frac{\partial H_z}{\partial x} + M_z \frac{\partial H_z}{\partial z} \right) \hat{k}. \tag{11}$$

Therefore Equation (5) becomes,

$$-\nabla p + \mu \frac{d^2 u}{dz^2}\hat{i} + \mu_0 \left(M_x \frac{\partial H_x}{\partial x} + M_z \frac{\partial H_x}{\partial z} \right) \hat{i} + \mu_0 \left(M_x \frac{\partial H_z}{\partial x} + M_z \frac{\partial H_z}{\partial z} \right) \hat{k} = 0,$$
$$\mu \frac{d^2 u}{dz^2}\hat{i} = \frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k} - \mu_0 \left(M_x \frac{\partial H_x}{\partial x} + M_z \frac{\partial H_x}{\partial z} \right) \hat{i} - \mu_0 \left(M_x \frac{\partial H_z}{\partial x} + M_z \frac{\partial H_z}{\partial z} \right) \hat{k}, \tag{12}$$

Equation (12) in component form is given as,

$$-\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial z^2} + \mu_0 \left(M_x \frac{\partial H_x}{\partial x} + M_z \frac{\partial H_x}{\partial z} \right) = 0, \tag{13}$$

$$-\frac{\partial p}{\partial z} + \mu_0 \left(M_x \frac{\partial H_z}{\partial x} + M_z \frac{\partial H_z}{\partial z} \right) = 0. \tag{14}$$

The continuity equation for an incompressible fluid is

$$\nabla \cdot \vec{q} = 0. \tag{15}$$

The Maxwell's equations [4, 11] are given by,

$$\nabla \times \vec{H} = 0, \quad \vec{H} = -\nabla \phi, \ \nabla \cdot \vec{B} = 0, \quad \vec{B} = \mu_0(\vec{H} + \vec{M}). \tag{16}$$

For ferrofluid the magnetization equation [12, 13]is given by,

$$\vec{M} = \overline{\mu}\vec{H}. \tag{17}$$

Nomenclature: $\vec{q}$ is the velocity of the fluid, ρ is the density of the fluid, $\vec{g}$ is the gravitation, P is the pressure, $\vec{M}$ is the magnetization, $\vec{H}$ is the magnetic field intensity, μ_0 is the permeability of free space, $\overline{\mu}$ is the magnetic susceptibility, η is the coefficient of viscosity of the fluid.

3   Method of Solution

From equation (16) we have,

$$H_x = -\frac{\partial \phi}{\partial x} \quad H_z = -\frac{\partial \phi}{\partial z}, \tag{18}$$

$$M_x = \overline{\mu} H_x \quad M_z = \overline{\mu} H_z, \tag{19}$$

$$\vec{H}^2 = H_x{}^2 + H_y{}^2, \tag{20}$$

Consider,

$$M_x \frac{\partial H_x}{\partial x} + M_z \frac{\partial H_x}{\partial z} = \overline{\mu} H_x \frac{\partial}{\partial x} \left(-\frac{\partial \phi}{\partial x} \right) + \overline{\mu} H_z \frac{\partial}{\partial z} \left(-\frac{\partial \phi}{\partial x} \right), \tag{21}$$

using equations (7), (8) and (9) in equation (21), we get

$$M_x \frac{\partial H_x}{\partial x} + M_z \frac{\partial H_x}{\partial z} = \overline{\mu} \left(-\frac{\partial \phi}{\partial x} \right) \frac{\partial}{\partial x} \left(-\frac{\partial \phi}{\partial x} \right) + \overline{\mu} \left(-\frac{\partial \phi}{\partial z} \right) \frac{\partial}{\partial z} \left(-\frac{\partial \phi}{\partial x} \right), \tag{22}$$

$$M_x \frac{\partial H_x}{\partial x} + M_z \frac{\partial H_x}{\partial z} = \overline{\mu} \left(\frac{\partial \phi}{\partial x} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial \phi}{\partial z} \frac{\partial^2 \phi}{\partial x \partial z} \right), \tag{23}$$

$$M_x \frac{\partial H_x}{\partial x} + M_z \frac{\partial H_x}{\partial z} = \frac{\overline{\mu}}{2} \frac{\partial}{\partial x} \left\{ \left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial z} \right)^2 \right\}, \tag{24}$$

$$M_x \frac{\partial H_x}{\partial x} + M_z \frac{\partial H_x}{\partial z} = \frac{\overline{\mu}}{2} \frac{\partial}{\partial x} (H_x{}^2 + H_y{}^2), \tag{25}$$

$$M_x \frac{\partial H_x}{\partial x} + M_z \frac{\partial H_x}{\partial z} = \frac{\overline{\mu}}{2} \frac{\partial}{\partial x} (\vec{H}^2). \tag{26}$$

Also consider,

$$M_x \frac{\partial H_z}{\partial x} + M_z \frac{\partial H_z}{\partial z} = \overline{\mu} \vec{H}_x \frac{\partial}{\partial x} \left(-\frac{\partial \phi}{\partial z} \right) + \overline{\mu} \vec{H}_z \frac{\partial}{\partial z} \left(-\frac{\partial \phi}{\partial z} \right), \tag{27}$$

Using equations (18), (19) and (20) in equation (27), we get

$$M_x \frac{\partial H_z}{\partial x} + M_z \frac{\partial H_z}{\partial z} = \overline{\mu} \left(-\frac{\partial \phi}{\partial x} \right) \frac{\partial}{\partial x} \left(-\frac{\partial \phi}{\partial z} \right) + \overline{\mu} \left(-\frac{\partial \phi}{\partial z} \right) \frac{\partial}{\partial z} \left(-\frac{\partial \phi}{\partial z} \right), \tag{28}$$

$$M_x \frac{\partial H_x}{\partial x} + M_z \frac{\partial H_x}{\partial z} = \frac{\overline{\mu}}{2} \frac{\partial}{\partial z} \left\{ \left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial z} \right)^2 \right\}, \tag{29}$$

$$M_x \frac{\partial H_x}{\partial x} + M_z \frac{\partial H_x}{\partial z} = \frac{\overline{\mu}}{2} \frac{\partial}{\partial z} (H_x{}^2 + H_y{}^2), \tag{30}$$

$$M_x \frac{\partial H_x}{\partial x} + M_z \frac{\partial H_x}{\partial z} = \frac{\overline{\mu}}{2} \frac{\partial}{\partial z} (\vec{H}^2). \tag{31}$$

Substituting equations (18) and (19) in equations (13) and (14), we get

$$-\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial z^2} + \frac{\mu_0 \overline{\mu}}{2} \frac{\partial}{\partial x} (\vec{H}^2) = 0. \tag{32}$$

$$-\frac{\partial p}{\partial z} + \frac{\mu_0 \overline{\mu}}{2} \frac{\partial}{\partial z} (\vec{H}^2) = 0. \tag{33}$$

Since the pressure gradient is zero,from equation (32) and (33) we get

$$\frac{\partial^2 u}{\partial z^2} = -\frac{\mu_0 \overline{\mu}}{2\mu} \frac{\partial}{\partial x}(\vec{H}^2), \tag{34}$$

Integrating equation (34), we get

$$u = Az + B, \tag{35}$$

where A and B are constants to be determined.

Boundary Conditions:
    Since the fluid is viscous and the upper rigid plate is moving with constant speed U while the covers rigid plate is stationary, the no slip condition suggests that [14]

$$u = 0 \quad \text{at} \quad z = 0, \tag{36}$$

$$u = U \quad \text{at} \quad z = h, \tag{37}$$

using equation(36)and (37) in equation(35) we obtain $B = 0, \quad A = \frac{U}{h}$.
Hence equation (35) becomes,

$$u = \frac{Uz}{h}, \tag{38}$$

where, $u^* = \frac{u}{U}$ the non-dimensional velocity, $z^* = \frac{z}{h}$ the non-dimensional vertical length. Neglecting the (*) for simplicity, we get

$$u = z. \tag{39}$$

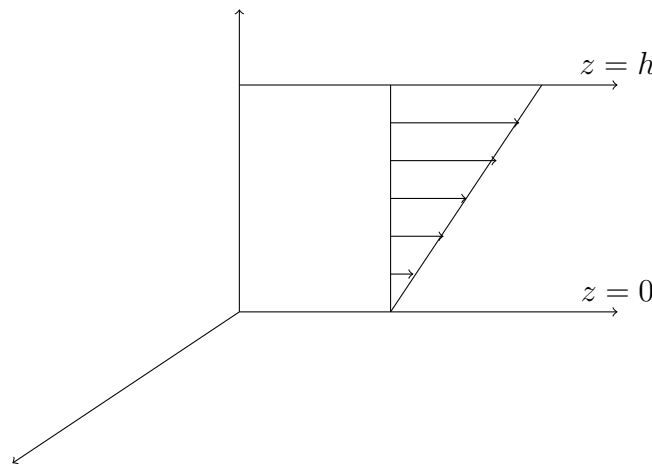Equation (39) is the velocity distribution for plane Couette flow which is shown in the following figure

Figure 2: Plane Couette flow

3.1   Maximum Velocity

$$u_{max} = z_{max}. \tag{40}$$

3.2 Average Velocity

The average velocity is defined as the ratio of the total flow over a cross section to the area of the cross section and is denoted by $\overline{u}$ [6, 10].

$$\overline{u} = \frac{1}{h} \int_0^h u \, dz, \tag{41}$$

$$\overline{u} = \frac{1}{h} \int_0^1 u^* U h \, dz^*, \tag{42}$$

$$\overline{u}^* = \int_0^1 u^* \, dz^*, \tag{43}$$

where, $\overline{u}^* = \frac{\overline{u}}{U}$ is the non-dimensional average velocity.
Neglecting asterisks $(*)$ for simplicity and integrating, we get

$$\overline{u} = \int_0^1 u \, dz, \tag{44}$$

$$\overline{u} = \frac{1}{2}. \tag{45}$$

3.3 Mass Flow Rate

The mass flow rate is defined as the amount of fluid that passes through any cross section of the channel per unit width and per unit time and is denoted by M [6, 10].

$$M = \rho \int_0^h u \, dz, \tag{46}$$

$$M = \rho h \left(\frac{1}{h} \int_0^h u \, dz \right), \tag{47}$$

$$M = \rho h \overline{u}, \tag{48}$$

$$M = \frac{\rho h}{2}. \tag{49}$$

4 Results and Discussions

Average velocity and mass flow rate are determined for ferrofluid in the absence of pressure gradient. We found that the average velocity is constant, whereas mass flow rate is proportional to the density of the fluid.

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Graeffe's Root Squaring Method For Nonlinear Equations

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Abstract: *The aim of this paper is to find the real, complex and distinct co-efficient of nonlinear polynomials using Graeffe's root squaring method. It is a direct method because it does not require any initial guess for roots. Here, we will give some applications of Graeffe's method and also we will solve few nonlinear polynomials using Graeffe's root squaring method.*

Keywords: *Graeffe's root squaring method, Nonlinear equations, Distinct roots, Equal roots, Complex roots.*

AMS Subject Classification: *65H04.*

1   Introduction

Germinal Pierre Dandelin was born in Paris and studying the Ecole poly technique. He was a professor of mining engineering in Belgium. He is the eponym of the Dandelin sphere of Dandelins theorem in geometry and of the Dandelin-Graeffe numerical method of solution of algebraic equations. He also published on the stereographic projection, algebraic, probability theorem.

In 1837, Karl Heinrich Graeffe also discovered the principal idea of the method. He was a German mathematician, who was at university of Zurich. The method separates the roots of a polynomial by squaring them repeatedly. This squaring of the roots is done implicitly, that is, only working on the coefficients of the polynomial. In 1828 he was appointed as the professor of the Zurich Institute of Technology and as of 1833, associate professor. Graeffe is known for having been the first to enunciate a method to approximate the roots of any polynomial, a method known today as the Dandelin-Graeffe method [1].

The only really useful practical method for solving numerical algebraic equations of higher orders, possessing complex roots, is that devised by C. H. Graeffe early in the nineteenth century. When an equation with real coefficients has only one or two pairs of complex roots, the Graeffe process leads to the evaluation of these roots without great labour. The classical root squaring method for approximating the real or complex roots of algebraic equations, usually called the Graeffe method. This is familiar to every numerical analyst and accounts of the procedure are in every textbook on numerical methods. The method works best when all the roots are real. The use of logarithm tables to obtain the absolute values of roots and other special tricks to get the complex roots are understandably enough, the consequences of hand calculation limitations.

1.1   Graeffe's root squaring method

Graeffe's method is one of the root finding method of a polynomial with real coefficients. This method gives all the roots approximated in each iteration. Also this is one of the direct

root finding method. Because this method does not requires any initial guesses for roots. A root finding method which was among the most popular methods for finding roots of univariate polynomial in the 19th and 20th centuries.

Graeffe's method has no first approximation need to be known. It has approximations to all roots are obtained simultaneously, in contradiction to the other methods which furnish approximations to one root at a time. In spite of this, the computations required by Graeffe's method are not much more laborious than those necessary to obtain an approximation to a single root by one of the other methods if allowance is made for the time necessary to find the first approximation. Its main advantage is that it also affords a means of finding the complex roots. It is true that by certain other methods. These advantages are well known, though not sufficiently appreciated in practice. But so for as can be seen, it is not known that Graeffe's method also gives the multiple roots, in a manner essentially simpler than is generally pointed out in the more elaborate descriptions of the procedure. Further, of all the methods it is the only one for solving an equation having several pairs of complex roots of the same modulus [1].

2   Applications of Graeffe's root squaring method

1. Graeffe's root square method is applicable in aerodynamics and circuit analysis.
2. Graeffe's root square method is applicable for solving numerical algebraic equations of higher order, possessing complex roots. when an equation with real coefficient has only one or pair of complex roots, the Graeffe process leads to the evaluation of these roots without any great labor.
3. Graeffe's root square method is applicable to solve the numerical algebraic equations [2].

3   Method of solution

3.1   Distinct roots

Consider the polynomial,

$$p_n(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n. \tag{1}$$

By separating the even power of x and odd power of x in equation (1) we get,

$$(a_0 x^n + a_2 x^{n-2} + a_4 x^{n-4} + \cdots) = (a_1 x^{n-1} + a_3 x^{n-3} + a_5 x^{n-5} + \cdots).$$

Squaring on both sides we obtain,

$$(a_0 x^n + a_2 x^{n-2} + a_4 x^{n-4} + \cdots)^2 = (a_1 x^{n-1} + a_3 x^{n-3} + a_5 x^{n-5} + \cdots)^2$$

$$a_0^2 x^{2n} - (a_1^2 - 2a_0 a_2) x^{2(n-1)} + (a_2^2 - 2a_1 a_3 + 2a_0 a_4) x^{2(n-2)} \mp \cdots + (-1)^n a_n^2 = 0. \tag{2}$$

Put $z = -x^2$ in equation (2) we have,

$$b_0 z^n + b_1 z^{n-1} + b_2 z^{n-2} + \cdots + b_{n-1} + b_n, \tag{3}$$

where,

$$b_0 = a_0^2$$
$$b_1 = a_1^2 - 2a_0a_2$$
$$b_2 = a_2^2 - 2a_1a_3 + 2a_0a_4$$
$$\vdots$$
$$b_n = a_n^2$$

Thus all the $b_k's$ $(k = 0, 1, 2, \cdots, n)$ are known in terms of $a_k's$ and $-\xi_1^2, -\xi_2^2, \cdots, -\xi_n^2$ are the roots of equation (3) and $\xi_1, \xi_2, \cdots, \xi_n$ are roots of equation (1). Then the coefficients of $b_k's$ are obtain as follows,

a_0	a_1	a_2	a_3	...	a_n
	a_1^2	a_2^2	a_3^2	...	a_n^2
	$-2a_0a_2$	$-2a_0a_3$	$-2a_2a_4$	...	
		$2a_0a_4$	$2a_1a_5$	...	
			$\cdot$	$\cdot$	
			$\cdot$	$\cdot$	
			$\cdot$	$\cdot$	
b_0	b_1	b_2	b_3	...	b_n

The first terms in the above table starting with positive signs then it having alternate signs. The first term is the squaring of the coefficients a_k. The second term is twice the product of the nearest neighboring coefficient a_{k-1} and a_{k+1}. The next neighboring coefficient a_{k-2} and a_{k+1} is also twice the product of the third term. This process is continued until there are no available coefficients to forms the cross product. This procedure is repeated many times so that the final equation is,

$$B_0x^n + B_1x^{n-1} + B_2x^{n-2} + \cdots + B_{n-1}x + B_n = 0. \tag{4}$$

And equation (4) has the roots $R_1, R_2, \cdots, R_n$ such that $R_i = -\xi_i^{2m}$, for $i = 1, 2, \cdots, n$.

Suppose $|\xi_1| > |\xi_2| > |\xi_3| > \cdots |\xi_n|$ then $|R_1| >> |R_2| >> \cdots >> |R_n|$. Hence the roots R_i are very widely separated for large m and we have,

$$-\frac{B_1}{B_0} = \sum R_i \approx R_1$$
$$-\frac{B_2}{B_0} = \sum R_iR_j \approx R_1R_2$$
$$-\frac{B_3}{B_0} = \sum R_iR_jR_k \approx R_1R_2R_3$$
$$\vdots$$
$$(-1)^n\frac{B_n}{B_0} = R_1R_2\cdots R_n$$
$$R_i = -\frac{B_i}{B_{i-1}}, i = 1, 2, 3 \cdots, n$$

Applying magnitude on both sides we get,

$$\xi_i^{2m} = |R_i| = \frac{B_i}{B_{i-1}}, i = 1, 2, 3, \cdots, n. \tag{5}$$

The equation (5) determines the absolute values of the roots and substituting this in equation (1) will gives the sign of the roots [3].

3.2   Equal roots

If two roots of an equation are numerically equal, the root-squares process can never breakup the equation. After that the magnitude of the B_k is half of the square of the magnitude of the corresponding coefficient then it indicates that ξ_k is double roots.

$$R_k \approx -\frac{B_k}{B_{k-1}} \text{ and } R_{k+1} = -\frac{B_{k+1}}{B_k}$$

$$R_k R_{k+1} \approx R_k^2 \approx \left| \frac{B_{k+1}}{B_{k-1}} \right|$$

$$|R_k|^2 \approx |\xi_k|^{2(2^m)} = \left| \frac{B_{k+1}}{B_{k-1}} \right|$$

This gives the magnitude of the double root. Directly we have to found the double roots. Since R_k and R_{k+1} converges to the same root after sufficient squaring this converges to the double root is slow. It would require few more squaring to stabilize the roots [4].

3.3   Complex roots

If the roots of ξ_k and ξ_{k+1} form a complex pairs, then this would cause the coefficients of x^{n-k} in this successive squaring to fluctuate both in magnitude and sign. If $\xi_k, \xi_{k+1} = \beta_k \exp(\pm iQ_k)$ is the complex pairs and we applying the magnitude and we get the real part,

$$2\beta_k^m \cos(m\Phi_k)$$

For m sufficiently large, β_k can be determined from the below relation,

$$|\beta_k|^{2(2^m)} \approx \left| \frac{B_{k+1}}{B_{k-1}} \right|$$

and ϕ is suitably determined from the relation,

$$2\beta_k^m \cos(m\Phi_k) \approx \left| \frac{B_{k+1}}{B_{k-1}} \right|.$$

If the equation has only on complex pair, then we can first determine all the roots. The complex pair can be written as $\xi_k, \xi_{k+1} = p \pm iq$. The sum of the roots is given by, [3]

$$\xi_1 + \xi_2 + \cdots + \xi_{k-1} + 2p + \xi_{k+2} + \cdots + \xi_n = \frac{-a_1}{a_0}$$

and p is determined by, $|\beta_k|^2 = p^2 + q^2.$

4   Examples and Discussions

Example 1: Find all the roots of the polynomial $x^3 - 6x^2 + 11x - 6 = 0$ using the Graeffe's root squaring method [5].

Solution: Given, $x^3 - 6x^2 + 11x - 6 = 0$

The coefficients of the successive root squaring are tabulated in the below table,

m	2^m	a_0	a_1	a_2	a_3
0	1	1	-6	11	-6
		1	36	121	36
			-22	-72	
1	2	1	14	49	36
		1	196	2401	1296
			-98	-1008	
2	4	1	98	1393	1296
		1	9604	1940449	1679616
			-2786	-254016	
3	8	1	6818	1686433	1679616
		1	46485124	2.8440563×10^{12}	$2.821109907 \times 10^{12}$
			-3372866	$-2.2903243 \times 10^{10}$	
4	16	1	43112258	2.8211530×10^{12}	$2.82110990x \times 10^{12}$

We know that,

$$|\xi_i|^{2^m} = |R_i| = \left| \frac{B_i}{B_{i-1}} \right|$$

For $i = 1$

$$|\xi_1|^{16} = \left| \frac{B_1}{B_0} \right| = \left| \frac{43112258}{1} \right| = |43112258|$$

$$|\xi_1| = |43112258|^{\frac{1}{16}} = 3.00028$$

For $i = 2$

$$|\xi_2|^{16} = \left| \frac{B_2}{B_1} \right| = \left| \frac{2.8211530 \times 10^{12}}{43112258} \right| = |654773044|$$

$$|\xi_2| = |654773044|^{\frac{1}{16}} = 1.999 \approx 2$$

For $i = 3$

$$|\xi_3|^{16} = \left| \frac{B_3}{B_2} \right| = \left| \frac{2.821109907 \times 10^{12}}{2.8211530 \times 10^{12}} \right| = |0.99984672|$$

$$|\xi_3| = |0.99984672|^{\frac{1}{16}} = 0.9999 \approx 1$$

The exact roots of the equations are $\xi_1 = 3$, $\xi_2 = 2$, $\xi_3 = 1$.

Example 2: Find all the roots of the polynomial $x^3 - 4x^2 + 5x - 2 = 0$ using the Graeffe's root squaring method.

Solution: Given, $x^3 - 4x^2 + 5x - 2 = 0$

The coefficients of the successive root squaring are tabulated in the below table,

m	2^m	a_0	a_1	a_2	a_3
0	1	1	-4	5	-2
		1	16	25	4
			-10	-16	
1	2	1	6	9	4
		1	36	81	16
			-18	-48	
2	4	1	18	33	16
		1	324	1089	256
			-66	-573	
3	8	1	258	513	256
		1	66564	263169	65536
			-1026	132096	
4	16	1	65538	131073	65536
		1	4295229444	$1.71801313 \times 10^{10}$	4294967296
			262146	8590196736	
5	32	1	4294967298	8589934594	4294967296

In this table we notice that the magnitude of the coefficient B_2 is half of the square of the magnitude of the corresponding coefficient. This phenomenon indicates that ξ_2 is a double root. We obtained the magnitude of the root as,

$$|\xi_k|^{2^m} = \left| \frac{B_{k+1}}{B_{k-1}} \right|$$

For $k = 1$
$$|\xi_1|^{2^m} = \left| \frac{B_2}{B_0} \right|$$

$$|\xi_1|^{32} = \left| \frac{8589934594}{1} \right|$$

$$|\xi_1| = |8589934594|^{\frac{1}{32}} = 1.9999$$

Hear $|\xi_2| = B_2$ is double root.
For $k = 2$
$$|\xi_2|^{64} = \left| \frac{B_3}{B_1} \right| = \left| \frac{4294967296}{4294967298} \right| = |0.9999|$$

$$|\xi_2| = |0.9999|^{\frac{1}{64}} = 1 = \xi_3.$$

All the roots are positive and the exact roots are $2, 1, 1$.

Example 3: Find all the roots of the polynomial $x^4 - x^3 + 3x^2 + x - 4 = 0$ using the Graeffe's root squaring method.

Solution: Given, $x^4 - x^3 + 3x^2 + x - 4 = 0$

The coefficients of the successive root squaring are tabulated in the below table,

m	2^m	a_0	a_1	a_2	a_3	a_4
0	1	1	-1	3	1	-4
		1	1	9	1	16
			-6	2	24	
				-8		
1	2	1	-5	3	25	16
		1	25	9	625	256
			-6	250	-96	
				32		
2	4	1	19	291	529	2656
		1	361	84681	279841	65536
			-582	-20102	148992	
				512		
3	8	1	-221	65091	130849	65536

Since B_1 alternates in signs, we have a pair of complex roots which can be obtained using B_0, B_1 and B_2. By real roots we have,

$$|\alpha| = |R_i| = \left| \frac{B_i}{B_{i-1}} \right| = |\xi^{2^m}|$$

First real root

$$|\alpha_1|^8 = \left| \frac{B_4}{B_3} \right| = \left| \frac{65536}{130894} \right| = |0.500852127|$$

$$|\alpha_1| = (|0.500852178|)^{\frac{1}{8}}$$

$$|\alpha_1| = |\xi_1| = |0.917199| \approx 0.9172$$

By substituting the $|\xi_1|$ in given equation this root is found to be negative $\xi_1 = -0.9172$

Second real root

$$|\alpha_2|^8 = \left| \frac{B_3}{B_2} \right| = \left| \frac{130849}{65091} \right| = |2.010247192|$$

$$|\alpha_2| = |2.010247192|^{\frac{1}{8}}$$

$$|\alpha_2| = |\xi_2| = |1.0912| \approx 1.0912$$

This root is found to be positive.
To obtained the pair of complex roots $p \pm iq$, we have $\beta = \sqrt{p^2 + q^2}$,

where

$$|\beta|^{2^m} = \left| \frac{B_{k+1}}{B_{k-1}} \right|$$

Put $k = 1$

$$|\beta|^{16} = \left| \frac{B_2}{B_0} \right| = \left| \frac{65091}{1} \right|$$

$$\beta = |65091|^{\frac{1}{16}} = 1.9991$$

We have

$$2p + \xi_1 + \xi_2 = \frac{-a_1}{a_0}$$

$$p = 0.413$$

and

$$q = \sqrt{\beta^2 - p^2} = \sqrt{(1.9991)^2 - (0.413)^2} = 1.953$$

Hence the root of the given polynomial are $-0.9172, 1.0912, 0.4130, \pm 1.9560i$.

5   Conclusion

In this chapter we discussed about Graeffe's root squaring method which is direct method, it yields not only one root at a time but also all roots (real, complex, equal) simultaneously. There is considerable saving in time also. In this method, there must be at least one real roots because complex roots occurs in pairs. Except Graeffe's root squaring method all other method requires a first approximation but the finding of the first approximation is difficult particularly in the case of complex roots. Graeffe's method on the other hand does not require a first approximate value. Besides, it is not necessary to use criteria of convergence in order to examine whether the approximate value of polynomial is sufficiently close to the actual roots. Several numerical examples are solved to illustrate the efficiency and the performance of the Graeffe's root squaring method . Therefore it seems to us that Graeffe's root squaring method is best for solving non linear algebraic equations.

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Approximation Theory using Trigonometric Polynomials

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Abstract: This article consists some examples on non linear equations which are solving by trigonometric polynomial approximation. In this article we discussed two methods such as continuous least square approximation and discrete least square approximation to find the polynomial for specified intervals.

Keywords: Approximation theory, Trigonometric polynomial, Fourier series, Orthogonal trigonometric polynomial, Discrete trigonometric approximation.

AMS Subject Classification: 42A10.

1 Introduction

The Approximation theory is the branch of mathematical analysis and studying numerical methods. It is concerned with how functions can be approximated with simple functions and quantitatively characterizing the errors. Approximation theory leads to approximate the given quantities by specific appropriate conditions for study about the size and properties. It is based upon summation of a series of terms with orthogonal polynomials and generalized Fourier series. They are obtained by power series expansions in which the higher order terms are dropped.

The main objective of approximation theory are the approximation of functions. Its foundations are laid by the work of P. L. Chebyshev (1854-1859) on uniform approximation of functions by polynomials and by K. Weierstrass, who in 1885 established the principle which is possible to approximate a continuous function on a finite interval by polynomials. The development of approximation theory was to a large extent determined by the fundamental work of H. Lebesgue, Ch. J. de la Vallee-Poussin, S.N. Bernstein, D. Jackson have done works on the approximation of functions. It also studies about the process of approximating general functions by simple functions such as polynomials, finite element or Fourier series.

Approximation theory plays a major role in the analysis of numerical methods. The approximation theory will explore different choices of approximation spaces and how they can be effective in different polynomials such as Trigonometric polynomials, Adomian polynomials, Least square approximation, Rational approximation, Chebyshev approximation etc [1].

1.1 Trigonometric polynomial

A Trigonometric polynomial is a finite linear combination of functions $\sin(nx)$ and $\cos(nx)$ with n taking on the values of one or more natural numbers. The use of series of sine and cosine functions to represent arbitrary functions had its beginnings in the 1750s with the study of the motion of a vibrating string. This problem was considered by Jean d'Alembert and the taken up by the foremost mathematician of the time, Leonhard Euler. Daniel Bernoulli was the one who first advocated the use of infinite sum of sine and cosine as a solution to the problem [2].

Polynomial is an expression which consists of operation such as addition, subtraction, multiplication and non-negative integer with exponents, variable and coefficient. Trigonometry table was apparently compiled by Hipparchus. He is known as father of trigonometry. Trigonometric polynomial is a function denoted as $T : \mathbb{R} \to \mathbb{R}$ with,

$$T(x) = \frac{a_0}{2} + \sum_{j=1}^{n} a_j \cos(jx) + b_j \sin(jx).$$

Fourier series is a trigonometric polynomial of order m with function of t is in the form of,

$$p(t) = a_0 + \sum_{k=1}^{m} (a_k \cos(kt) + b_k \sin(kt)),$$

where $a_0, a_1, a_2 \ldots a_m, b_1, b_2, \ldots b_m$ are coefficient of real numbers. Trigonometric polynomial in space is presented based on the basis, symmetric trigonometric polynomial approximation like Bernstein polynomial are constructed two kinds of nodes are given to show that the trigonometric polynomial sequence is uniformly convergent. Trigonometric polynomials will display the order of the polynomial and in the set of polynomials which are sum of multiples of $\sin(kx)$ and $\cos(kx)$ [3].

1.2   Fourier series

Fourier series is the sum of orthogonal sinusoidal, with representation of periodic function. Fourier series is introduced by, Jean Baptiste Joseph Fourier has made major contribution to study of trigonometric series. The product of two function with the weight function is equal to zero on the specified interval then the two functions are orthogonal. Fourier analysis depend on the choice of trigonometric function like sine and cosine, it is a central tool for periodic signals. Orthogonal polynomial is a another orthogonal family its find an application in numerical techniques such as least-square approximation. Orthogonality is very useful when we represent a periodic function as trigonometric series.

Vlamidir and Hans gave the approximation for continuous functions by using trigonometric polynomial in the interval $[-\pi, \pi]$. The trigonometric functions like cosine and sine defined for real number are continuous for all real numbers. The functions tangent, cotangent, secant, and co-secant are essential discontinuities. In general approximating polynomial include polynomial, trigonometric function, exponential function and rational function. When $p(x)$ is a polynomial which is approximated for any continuous function $f(x)$ on a finite interval $[a, b]$ is guaranteed by Weierstrass approximation theorem. In the least square approximation least squares means that the minimizing the overall solution to sum of square of the residues [4].

2   Applications of trigonometric polynomials

- The fundamental trigonometric functions like sine and cosine are used to describe the sound and light waves.
- It is used in the creation of Maps to calculate heights of waves and tides in ocean.
- It is used in geography to measure the distance between landmarks in astronomy to measure the distance of nearby stars and also in satellite navigation system gun shot.

- It is useful in fields like engineering physics surveyors architects even in the investigation of crime scene it is used to calculate a trajectory of a projectile and to causes of a collision in a car accident, further it is used to identify how an object falls in what angle.
- It is used in marine biology for measurements to figure out the depth of sunlight that affects algae to photosynthesis using trigonometric functions and it is used to estimate the size of larger animals like whales and also to understand their behavior
- It is used in developing computer music, as sound travels in the form of waves and this wave pattern through a sine or cosine function for developing computer music.
- It is used in measuring the height of a building or a mountain to find distance of building from view point and the elevation angle can easily determined the height of a building using trigonometric function.

3   Method of solution

Let $f(x)$ be a function a_0, a_n and b_n be the coefficient, cosine and sine be the trigonometric functions by taking the infinite sum of sine and cosine function for positive integers n. The Fourier series is constructed as,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx), \tag{1}$$

where, $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx,$ $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)dx,$ and $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)dx$

Then for the set of functions $\{\phi_0, \phi_1, \cdots, \phi_{2n-1}\}$. By using Fourier series we can write,

$$\phi_0(x) = \frac{1}{2}$$
$$\phi_k(x) = \cos kx \cdots \quad \forall\, k = 1, 2, \cdots n$$
$$\phi_{n+k}(x) = \sin kx \cdots \quad \forall\, k = 1, 2, \cdots n-1$$

The set is orthogonal on the interval $[-\pi, \pi]$ with respect to the weight function $[W(x)]$ is 1. The orthogonality for the integer j the integrals of $\sin(jx)$ and $\cos(jx)$ over $[-\pi, \pi]$ are 0. The product of sine and cosine can be written as the sum of trigonometric identities as fallows,

$$\sin t_1 \sin t_2 = \frac{1}{2}\left[\cos(t_1 - t_2) - \cos(t_1 + t_2)\right]$$
$$\cos t_1 \cos t_2 = \frac{1}{2}\left[\cos(t_1 - t_2) + \cos(t_1 + t_2)\right]$$
$$\sin t_1 \cos t_2 = \frac{1}{2}\left[\sin(t_1 - t_2) + \sin(t_1 + t_2)\right]$$

3.1   Orthogonal trigonometric polynomials

Let T_n is the set of all linear combination of functions $\phi_0, \phi_1, \cdots, \phi_{2n-1}$. The set includes set of linear combination of function such as $\sin(nx)$ and $\cos(nx)$ for natural numbers is called as trigonometric polynomial where degree less than or equal to n. For the function $f \in C[-\pi, \pi]$. To find the continuous least square approximation for the functions in T_n is in the form,

$$S_n(x) = \frac{a_0}{2} + a_n \cos(nx) + \sum_{k=1}^{n-1} (a_k \cos(kx) + b_k \sin(kx)). \tag{2}$$

Since $\{\phi_0, \phi_1, \cdots, \phi_{2n-1}\}$ is an orthogonal set of function on $[-\pi, \pi]$ with $W(x) = 1$, where W is an weight function on interval I. If $W(x) \geq 0 \, \forall x$ in I but $W(x) \neq 0$ on any sub-interval in I. The weight function is mainly to assign varying degrees of importance to approximations on certain portion of the interval.

For example, $W(x) = \dfrac{1}{\sqrt{1 - x^2}}$ and $\phi_0, \phi_1, \cdots, \phi_n$ is a set of linearly independent function on $[a, b]$ and W is a weight function on $[a, b]$ when $f \in C[a, b]$. The linear combination is denoted as, $p(x) = \displaystyle\sum_{k=0}^{n} a_k \phi_k(x)$ to minimize the error we take,

$$E = E(a_0 \ldots a_n) = \int_a^b W(x)[f(x) - \sum_{k=0}^{n} a_k \phi_k(x)]^2 dx.$$

Partially differentiating with respect to $a_j \quad \forall j = 0, 1, \cdots, n$ we get,

$$\frac{\partial E}{\partial a_j} = 2 \int_a^b W(x)[f(x) - \sum_{k=0}^{n} a_k \phi_k(x)]dx(\phi_j(x)).$$

The system of normal equation can be written as,

$$\int_a^b W(x)f(x)\phi_j(x)dx = \sum_{k=0}^{n} a_k \int_a^b w(x)\phi_k(x)\phi_j(x)dx, \quad \forall j = 0, 1, \cdots, n.$$

If the functions $\phi_0, \phi_1, \cdots, \phi_n$ can be chosen as,

$$\int_a^b W(x)\phi_k(x)\phi_j(x)dx = \begin{cases} 0, & \text{when } j \neq k \\ \alpha_j > 0, & \text{when } j = k \end{cases}$$

If we choose the functions $\phi_0 \, \phi_1 \ldots \phi_n$ then we have,

$$a_j = \frac{1}{\alpha_j} \int_a^b W(x)f(x)\phi_j dx.$$

Now the least square approximation is simplified for the chosen function $\phi_0 \, \phi_1 \ldots \phi_n$, which satisfy the orthogonality condition.

- Orthogonal set of function $\{\phi_0 \, \phi_1 \ldots \phi_n\}$ is orthonormal when

$$\int_a^b W(x)\phi_k(x)\phi_j(x)dx = \begin{cases} 0, & j \neq k \\ \alpha_j > 0, & j = k \end{cases} \tag{3}$$

- The least square approximation for orthogonal set of function such as $\{\phi_0 \ldots \phi_n\}$ on $[a, b]$ with respect to weight function W can be written as,

$$a_j = \frac{\int_a^b W(x)\phi_j(x)f(x)dx}{\int_a^b W(x)[\phi_j(x)]^2 dx} = \frac{1}{\alpha_j} \int_a^b W(x)\phi_j(x)f(x)dx. \tag{4}$$

For $[-\pi, \pi]$ with $W(x) = 1$ using in equation 4 we get,

$$a_k = \frac{\int_{-\pi}^{\pi} f(x)\cos(kx)dx}{\int_{-\pi}^{\pi} (\cos(kx))^2 dx} \tag{5}$$

Consider,

$$\int_{-\pi}^{\pi} (\cos(kx))^2 = \pi \tag{6}$$

By substituting equation (6) in equation (5) we get,

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(kx)dx \quad \forall\, k = 0, 1, 2, \ldots n.$$

Similarly we can do it for sine function then we have,

$$b_k = \frac{\int_{-\pi}^{\pi} f(x)\sin(kx)dx}{\int_{-\pi}^{\pi} (\sin(kx))^2 dx} \tag{7}$$

Consider,

$$\int_{-\pi}^{\pi} (\sin(kx))^2 = \pi \tag{8}$$

By substituting equation (8) in equation (7) we get,

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(kx)dx \quad \forall\, k = 0, 1, 2, \ldots n.$$

Then by substituting limit $n \to \infty$ in equation (2) then we get Fourier Series. It is used to describe the both ODE and PDE occur in physical situations [2].

3.2   Discrete trigonometric approximation

Discrete least square approximation and interpolation for large amount of data is done by using discrete analogue. For the collection of $2m$ paired data points $\{(x_j, y_j)\}_{j=0}^{2m-1}$ first element in this is equally partitioned in a closed interval let us choose it as $[-\pi, \pi]$.

$$x_j = -\pi + \left(\frac{j}{m}\right)\pi \quad \text{for each } j = 0, 1, 2 \ldots, 2m - 1$$

If the interval is not $[-\pi, \pi]$ then we have to apply a simple linear transformation to transform it into the $[-\pi, \pi]$. Discrete case is mainly used to determine the trigonometry polynomial $S_n(x)$ in T_n to minimize we can take,

$$E(S_n) = \sum_{j=0}^{2m-1} [y_j - S_n(x_j)]^2.$$

By substituting $S_n(x)$ in x_j form we get,

$$E(S_n) = \sum_{j=0}^{2m-1} \left\{ y_j - \left[\frac{a_0}{2} + a_n\cos(nx_j) + \sum_{k=1}^{n-1}(a_k\cos(kx_j) + b_k\sin(kx_j)) \right] \right\}^2.$$

To determine the constant for the equally spaced points $\{x_j\}_{j=0}^{2m-1}$ and summation in $[-\pi, \pi]$ is,

$$\sum_{j=0}^{2m-1} \phi_k(x_j)\phi_l(x_j) = 0 \text{ for } k \neq l$$

To check this orthogonality condition we need to prove the lemma.

3.2.1   Lemma

Suppose that the integer r is not a multiple of $2m$ then,

- $\displaystyle\sum_{j=0}^{2m-1} \cos(rx_j) = 0$ and $\displaystyle\sum_{j=0}^{2m-1} \sin(rx_j) = 0$

  Moreover if r is not a multiple of m,   then

- $\displaystyle\sum_{j=0}^{2m-1} (\cos(rx_j))^2 = m$ and $\displaystyle\sum_{j=0}^{2m-1} (\sin(rx_j))^2 = m$

Proof: let us consider the Euler formula to prove the Lemma.
Euler formula states that $i^2 = -1$ then for each real value z we can write it as,

$$e^{iz} = \cos(z) + i\sin(z) \tag{9}$$

Applying the summation to equation (9) we get,

$$\sum_{j=0}^{2m-1} \cos(rx_j) + i\sum_{j=0}^{2m-1} \sin(rx_j) = \sum_{j=0}^{2m-1} (\cos(rx_j) + i\sin(rx_j).$$

By using equation (9) we get,

$$\sum_{j=0}^{2m-1} \cos(rx_j) + i\sum_{j=0}^{2m-1} \sin(rx_j) = \sum_{j=0}^{2m-1} e^{irx_j}, \tag{10}$$

where $x_j = -\pi + \frac{j}{m}\pi$.
Considering e^{irx_j} and substituting x_j we get,

$$e^{irx_j} = e^{ir(-\pi+j(\pi/m))} = e^{-ir\pi} \cdot e^{(irj\pi)/(m)}. \tag{11}$$

By substituting equation (11) in equation (10) we get,

$$\sum_{j=0}^{2m-1} \cos(rx_j) + i\sum_{j=0}^{2m-1} \sin(rx_j) = e^{-ir\pi}\sum_{j=0}^{2m-1} e^{(irj\pi)/m}. \tag{12}$$

Since first term is 1 and ratio $e^{ir\pi/m} \neq 1$ and also $\displaystyle\sum_{j=0}^{2m-1} e^{(irj\pi)/m}$ is in geometric series. We get in the form of,

$$\sum_{j=0}^{2m-1} e^{(irj\pi)/m} = \frac{1 - \left(e^{ir\pi/m}\right)^{2m}}{1 - e^{ir\pi/m}} = \frac{1 - e^{2ir\pi}}{1 - e^{ir\pi/m}}. \tag{13}$$

But

$$e^{2ir\pi} = \cos(2\pi r) + i\sin(2\pi r) = 1 \text{ then } 1 - e^{2ir\pi} = 0. \tag{14}$$

By substituting equation (13) and equation (14) in equation (12) we get,

$$\sum_{j=0}^{2m-1} \cos(rx_j) + i\sum_{j=0}^{2m-1} \sin(rx_j) = 0.$$

Separating the imaginary and real parts we get,

$$\sum_{j=0}^{2m-1} \cos(rx_j) = 0 \text{ and } \sum_{j=0}^{2m-1} \sin(rx_j) = 0.$$

To prove for r is not multiple of m we have, $\sum_{j=0}^{2m-1} \cos(rx_j)^2 = m$ and $\sum_{j=0}^{2m-1} \sin(rx_j)^2 = m$

Hence lemma is proved for both the condition. Now let us prove the orthogonality condition by considering the example,

$$\sum_{j=0}^{2m-1} \phi_k(x_j)\phi_{n+l}(x_j) = \sum_{j=0}^{2m-1} \cos(kx_j)\sin(lx_j). \tag{15}$$

By substituting, $\qquad \cos(t_1)\sin(t_2) = \frac{1}{2}[\sin(t_1 - t_2) + \sin(t_1 + t_2)]$

We can write it as,

$$\sum_{j=0}^{2m-1} \cos(kx_j)\sin(lx_j) = \frac{1}{2}\left[\sum_{j=0}^{2m-1} \sin(l+k)x_j + \sum_{j=0}^{2m-1} \sin(l-k)x_j\right].$$

Since $(l+k)$ and $(l-k)$ are not multiple of $2m$ we get,

$$\sum_{j=0}^{2m-1} \cos(kx_j)\sin(lx_j) = 0.$$

3.2.2 Theorem

The constant in the summation

$$S_n(x) = \frac{a_0}{2} + a_n \cos(nx) + \sum_{k=1}^{n-1} (a_k \cos(kx) + b_k \sin(kx))$$

that minimize the least square sum as,

$$E(a_0 \ldots a_n, b_1 \ldots b_{n-1}) = \sum_{j=0}^{2m-1} [y_j - S_n(x_j)]^2$$

are

$$a_k = \frac{1}{m} \sum_{j=0}^{2m-1} y_j \cos(kx_j) \quad \forall\, k = 0, 1, 2, \ldots, n$$

and

$$b_k = \frac{1}{m} \sum_{j=0}^{2m-1} y_j \sin(kx_j) \quad \forall\, k = 1, 2, \ldots, n-1.$$

Proof: Given,

$$S_n(x) = \frac{a_0}{2} + a_n \cos(nx) + \sum_{k=1}^{n-1} (a_k \cos(kx) + b_k \sin(kx)) \tag{16}$$

$$E(a_0 \ldots a_n, b_1 \ldots b_{n-1}) = \sum_{j=0}^{2m-1} [y_j - S_n(x_j)]^2. \tag{17}$$

Let differentiate equation (17) partially with respect a_k and b_k and equate to zero.
Differentiating partially with respect to a_k we get,

$$\frac{\partial E}{\partial a_k} = \sum_{j=0}^{2m-1} y_j \cos(kx_j) - \sum_{j=0}^{2m-1} S_n(x_j) \cos(kx_j).$$

By substituting the equation (17) we get,

$$0 = \sum_{j=0}^{2m-1} y_j \cos(kx_j) - \frac{a_0}{2} \sum_{j=0}^{2m-1} \cos(kx_j) - a_n \sum_{j=0}^{2m-1} \cos(kx_j) \cos(nx_j)$$
$$- \sum_{l=1}^{n-1} a_l \sum_{j=0}^{2m-1} \cos(kx_j) \cos(lx_j) - \sum_{\substack{l=1 \\ l \neq k}}^{n-1} b_l \sum_{j=0}^{2m-1} \cos(kx_j) \sin(lx_j) - a_k \sum_{j=0}^{2m-1} (\cos(kx_j))^2. \tag{18}$$

We know that,

$$\sum_{j=0}^{2m-1} \cos(rx_j) = 0 \text{ and } \sum_{j=0}^{2m-1} \sin(rx_j) = 0 \tag{19}$$

$$\sum_{j=0}^{2m-1} \cos(rx_j)^2 = m \text{ and } \sum_{j=0}^{2m-1} \sin(rx_j)^2 = m \tag{20}$$

$$\sum_{j=0}^{2m-1} \cos(kx_j) \sin(lx_j) = 0. \tag{21}$$

By using equations (19) ,(20) and (21) in equation (18) we get,

$$a_k = \frac{1}{m} \sum_{j=0}^{2m-1} \cos(kx_j) y_j.$$

Similarly differentiating partially with respect to b_k we obtain,

$$\frac{\partial E}{\partial b_k} = \sum_{j=0}^{2m-1} y_j \sin(kx_j) - \sum_{j=0}^{2m-1} S_n(x_j) \sin(kx_j).$$

By substituting the equation (16) we get,

$$0 = \sum_{j=0}^{2m-1} y_j \sin(kx_j) - \frac{a_0}{2} \sum_{j=0}^{2m-1} \sin(kx_j) - a_n \sum_{j=0}^{2m-1} \sin(kx_j) \cos(nx_j) - \sum_{l=1}^{n-1} a_l \sum_{j=0}^{2m-1} \sin(kx_j) \cos(lx_j)$$

$$-\sum_{\substack{l=1\\l\neq k}}^{n-1} b_l \sum_{j=0}^{2m-1} \sin(kx_j)\sin(lx_j) - b_k \sum_{j=0}^{2m-1} (\sin(kx_j))^2. \tag{22}$$

Also by using equation (19) ,(20) and (21) in equation (22) we get,

$$b_k = \frac{1}{m} \sum_{j=0}^{2m-1} \sin(kx_j)y_j.$$

Hence proved the theorem [2].

4   Examples and Discussions

Example 1: Show that the function $\phi_0(x) = 1/2$, $\phi_1(x) = \cos(x) \ldots, \phi_n(x) = \cos(nx)$, $\phi_{n+1}(x) = \sin(x) \ldots \phi_{2n-1}(x) = \sin(n-1)(x)$ are orthogonal on $[-\pi, \pi]$ with respect to $W(x) = 1$.

Solution: Given

$$\phi_0(x) = 1/2$$
$$\phi_i(x) = \cos(ix) \quad \forall \; i = 1 \ldots n$$
$$\phi_{n+1}(x) = \sin(ix) \quad \forall \, i = 1 \ldots n - 1$$

To show that the given functions are orthogonal on $[-\pi, \pi]$ with respect to $W(x) = 1$.
We know that

$$2\sin(A)\cos(B) = \sin\left(\frac{A+B}{2}\right) + \sin\left(\frac{A-B}{2}\right).$$

Case 1: For $i = 1, 2, \ldots n$
Since cosine is even

$$\int_{-\pi}^{\pi} W(x)\phi_0(x)\phi_i(x)dx = \left[\frac{\sin(ix)}{i}\right]_0^{\pi}$$

$$\int_{-\pi}^{\pi} W(x)\phi_0(x)\phi_i(x)dx = 0.$$

Case 2: For $i = 1, 2, \ldots n - 1$

$$\int_{-\pi}^{\pi} W(x)\phi_0(x)\phi_{n+1}(x)dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin(ix)dx$$

Since sine is odd function,

$$\int_{-\pi}^{\pi} W(x)\phi_0(x)\phi_{n+1}(x)dx = 0.$$

Case 3: For $i = 1, \ldots, n$ and $j = 1 \ldots n - 1$

$$\int_{-\pi}^{\pi} W(x)\phi_i(x)\phi_{n+j}(x)dx = \int_{-\pi}^{\pi} \cos(ix)\sin(jx)dx$$

Since sine is odd function,

$$\int_{-\pi}^{\pi} W(x)\phi_i(x)\phi_{n+j}(x)dx = 0.$$

Case 4:

$$\int_{-\pi}^{\pi} W(x)\phi_0(x)\phi_0(x)dx = \int_{-\pi}^{\pi} \frac{1}{4}dx$$

$$\int_{-\pi}^{\pi} W(x)\phi_0(x)\phi_0(x)dx = \frac{2\pi}{4} > 0$$

Case 5:

$$\int_{-\pi}^{\pi} W(x)\phi_i(x)\phi_i(x)dx = \int_{-\pi}^{\pi} \cos^2(ix)dx \quad \forall i = 1\ldots n$$

Since $\int_{-\pi}^{\pi} \cos^2(ix)dx > 0$, hence $\cos^2(ix) > 0$ on $(0, \pi/2)$ and non-negative on $[-\pi, \pi]$.

Case 6:

$$\int_{-\pi}^{\pi} W(x)\phi_{n+j}(x)\phi_{n+j}(x)dx = \int_{-\pi}^{\pi} \sin^2(jx)dx \quad \forall j = 1\ldots n-1$$

Since $\int_{-\pi}^{\pi} \sin^2(jx)dx > 0$, hence $\sin^2(jx) \geq 0$ on $[-\pi, \pi)$ and strictly positive on $[0, \pi]$.

From all the cases we have verified the definition of orthogonality with respect a weight function. This means given function orthogonal with respect weight function hence proved.

Example 2: Find the continuous least square trigonometric polynomial $S_3(x)$ for $f(x) = e^x$ on the interval $[-\pi, \pi]$.

Solution: Given. $f(x) = e^x$ for $[-\pi, \pi]$.
We know that,

$$S_n(x) = \frac{a_0}{2} + a_n \cos(nx) + \sum_{k=1}^{n-1}(a_k \cos(kx) + b_k \sin(kx)).$$

For $n = 3$ we get,

$$S_3(x) = \frac{a_0}{2} + a_3 \cos(3x) + \sum_{k=1}^{2}(a_k \cos(kx) + b_k \sin(kx)).$$

By expanding the summation we have,

$$S_3(x) = \frac{a_0}{2} + a_3 \cos(3x) + a_1 \cos(x) + a_2 \cos(2x) + b_1 \sin(x) + b_2 \sin(2x). \tag{23}$$

To find a_0, a_1, a_2, a_3, b_1 and b_2 we know that,

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)dx \quad \forall\ k = 0, 1, 2, \ldots n \tag{24}$$

and

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx \quad \forall \ k = 0, 1, 2, \ldots n. \tag{25}$$

By substituting $k = 0$ in equation (24) we obtain,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x dx = 7.3521558$$

For $k = 1$ in equation (24) we get,

$$a_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \cos(x) dx$$

Let,

$$I = \int_{-\pi}^{\pi} e^x \cos(x) dx$$

$$I = \frac{1}{2} [\cos(x) e^x + \sin(x) e^x]_{-\pi}^{\pi}$$

By substituting I value back to a_1 we get, $a_1 = -3.676077912$.
For $k = 2$ in equation (24) we get,

$$a_2 = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \cos(2x) dx$$

Let,

$$I = \int_{-\pi}^{\pi} e^x \cos(2x)$$

$$I = \frac{1}{5} [\cos(2x) e^x + 2\sin(2x) e^x]_{-\pi}^{\pi}$$

By substituting I back to the a_2 we obtain, $a_2 = 1.470431164$.
For $k = 3$ in equation (24) we get,

$$a_3 = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \cos(3x) dx$$

Let,

$$I = \int_{-\pi}^{\pi} e^x \cos(3x)$$

$$I = \frac{1}{10} [\cos(3x) e^x + 2\sin(3x) e^x]_{-\pi}^{\pi}$$

By substituting I back to the a_3 we get, $a_3 = -0.735215582$.
For $k = 1$ in equation (25) we obtain,

$$b_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \sin(x)$$

Let,

$$I = \int_{-\pi}^{\pi} e^x \sin(x)$$

$$I = \frac{1}{2} [\sin(x) e^x - \cos(x) e^x]_{-\pi}^{\pi}$$

Substituting I value back to the b_1 then, $b_1 = 3.676070$.
For $k = 2$ in equation (25) we get,
$$b_2 = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \sin(2x) dx$$

Let,
$$I = \int_{-\pi}^{\pi} e^x \sin(2x) dx$$

$$I = \frac{1}{5}[\sin(2x)e^x]_{-\pi}^{\pi} - 2\cos(2x)e^x]_{-\pi}^{\pi} - 4I]$$

Substituting I value back to the b_2 then, $b_2 = -2.940862$.
Substituting a_0, a_1, a_2, a_3, b_1 and b_2 in equation (23) we get,

$$S_3(x) = 3.67607791 - 3.676077912\cos(x) + 1.470431164\cos(2x) - 0.735215582\cos(3x)$$
$$+ 3.676070\sin(x) - 2.940862\sin(2x).$$

Example 3: Determine the error of discrete least square trigonometric polynomial $S_3(x)$ using $m = 4$ for $f(x) = e^x \cos(2x)$ on the interval $[-\pi, \pi]$.

Solution: Given $f(x) = e^x \cos(2x)$ when x is in $[-\pi, \pi]$ and $m = 4$ is the nodes
we know that $y_j = f(x_j) = e^{x_j}\cos(2x_j) \quad \forall\, j = 0, 1, 2, 3, 4, 5, 6, 7$.
where $x_j = -\pi + (\frac{j}{m})\pi$ and general trigonometric polynomial is,

$$S_n(x) = \frac{a_0}{2} + a_n\cos(nx) + \sum_{k=1}^{n-1}(a_k\cos(kx) + b_k\sin(kx)).$$

Put $n = 3$ we get,
$$S_3(x) = \frac{a_0}{2} + a_3\cos(3x) + \sum_{k=1}^{2}(a_k\cos(kx) + b_k\sin(kx)).$$

By expanding the summation we get,
$$S_3(x) = \frac{a_0}{2} + a_3\cos(3x) + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x). \tag{26}$$

To find the a_0, a_1, a_2, a_3, b_1 and b_2 we have,

$$a_k = \frac{1}{m}\sum_{j=0}^{2m-1} y_j\cos(kx_j) \quad \forall\, k = 1, 2, 3 \ldots n. \tag{27}$$

$$b_k = \frac{1}{m}\sum_{j=0}^{2m-1} y_j\sin(kx_j) \quad \forall\, k = 1, 2, 3 \ldots n. \tag{28}$$

Putting $k = 0, 1, 2,$ and 3 in equation (27) then we get,

$$a_0 = -0.4968928713$$
$$a_1 = 0.2391965$$
$$a_2 = 1.515392698$$
$$a_3 = 0.2391965$$

Putting $k = 1$ and 2 in equation (28) then we have,

$$b_1 = -1.15064947$$
$$b_2 = 0$$

By substituting a_0, a_1, a_2, a_3, b_1 and b_2 in equation (26) we get,

$$S_3(x) = -0.4968928713 + 0.2391965 \cos(x) + 1.515392698 \cos(2x)$$
$$+ 0.2391965 \cos(3x) - 1.15064947 \sin(x)$$

To find the error we have an formula such as,

$$E(S_3) = \sum_{j=0}^{2m-1} (y_j - S_3(x_j))^2,$$

where $y_j = f(x_j)$ and by expanding the summation we get,

$$E(S_3(x)) = [f(x_1)^2 + (S_3(x_1))^2 - 2(f(x_1)S_3(x_1))] + [f(x_2)^2 + (S_3(x_2))^2 - 2(f(x_2)S_3(x_2))]$$
$$+ [f(x_3)^2 + (S_3(x_3))^2 - 2(f(x_3)S_3(x_3))] + [f(x_4)^2 + (S_3(x_4))^2 - 2(f(x_4)S_3(x_4))]$$
$$+ [f(x_5)^2 + (S_3(x_5))^2 - 2(f(x_5)S_3(x_5))] + [f(x_6)^2 + (S_3(x_6))^2 - 2(f(x_6)S_3(x_6))]$$
$$+ [f(x_7)^2 + (S_3(x_7))^2 - 2(f(x_7)S_3(x_7))]$$
$$E(S_3(x)) = 7.2711.$$

5   Conclusion

In this article we discussed about trigonometric polynomial approximation method which is one of the very useful method to find the polynomial by using the given function and for the specified interval. By using the Fourier series expression deriving the formula for continuous and discrete least square approximation. We find the constant values and substituting back to the formula and also to find the errors of the given function. We also considered some examples to show trigonometric function is orthogonal with respect to weight function and for different intervals we need to apply the Fourier transform.

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Study of Hilbert Transforms and its Applications

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Abstract: *The Hilbert transform is one of the most important transformations or operators in the field of signal theory. In this paper we take a look at the definition of Hilbert transforms, its inverse transforms, examples and properties on Hilbert transforms and a few applications on boundary value problems.*

Keywords: *Integral transformations, Hilbert transforms, Inverse Hilbert transforms, Cauchy Principal Value, Properties of Hilbert transforms, Signal theory, Analytic signals, Applications on boundary value problem.*

AMS Subject Classification: *44A15.*

1 Introduction

The Hilbert transform, named after a German Mathematician David Hilbert, is one of the most important integral transformations in the field of signal theory. Leonard Euler, in 1973, gave the famous formula [1]

$$e^{ix} = \cos x + i \sin x \ \text{ for } \ x \in \mathbb{R}$$

Using the above equation, C.P. Steinmetz and A.E. Kennely introduced the complex notation of harmonic wave as

$$e^{i\omega t} = \cos \omega t + i \sin \omega t$$

But it was David Hilbert who showed that the function $\sin \omega t$ is the Hilbert transform of the function $\cos \omega t$, which led to the basic property of Hilbert transform, namely $\pm \frac{\pi}{2}$ phaseshift.

Its first use dates back to Hilbert's work concerning analytical function in connection to the Riemann problem. G.H. Hardy and E.C. Titchmarsh have also worked extensively on the properties of Hilbert transform and developed it.

Definition of Hilbert transform and inverse Hilbert transform

Consider a function $f(t)$ defined on the real line for $t \in (-\infty, \infty)$, then the Hilbert transform of $f(t)$, denoted by $\hat{f}_H(x)$, is defined as [2, 3]

$$H\{f(t)\} = \hat{f}_H(x) = \frac{1}{\pi} \oint_{-\infty}^{\infty} \frac{f(t)}{t - x} dt \tag{1}$$

But the integral is improper and the integrand has a singularity at $x = t$. Thus we make use of Cauchy Principal Value and define Hilbert transform of a function as follows

$$H\{f(t)\} = \frac{1}{\pi} \lim_{\epsilon \to 0} \left[\int_{\infty}^{x-\epsilon} \frac{f(t)}{t - x} dx + \int_{x+\epsilon}^{\infty} \frac{f(t)}{t - x} dx \right]$$

The Cauchy Principal Value is obtained by considering a finite range of integration which is symmetric about the point of singularity, but does not include the symmetric sub interval. We take the limit of the integral as the length of the interval approaches ∞ and the length of the excluded interval approaches 0 [4, 5, 6].

The inverse of Hilbert transform denoted by $H^{-1}\{\hat{f}_H(x)\}$ is defined as

$$f(t) = -\frac{1}{\pi} \int\limits_{-\infty}^{\infty} \frac{\hat{f}_H(x)}{x-t} dx \tag{2}$$

We can also observe $H^{-1} = -H$, in other words the inverse Hilbert transform of $f(t)$ is obtained by applying the Hilbert transform to $\hat{f}_H(x)$ and neglecting the result.

2   Properties of Hilbert Transforms

Theorem 1 (Linearity Property). *Let $f_1(t)$ and $f_2(t)$ be two function then*

$$H\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 H\{f_1(t)\} + c_2 H\{f_2(t)\}$$

where c_1, c_2 are constants. [7]

Proof. By using the Cauchy Principal Value definition of the Hilbert transform, we have

$$H\{c_1 f_1(t) + c_2 f_2(t)\} = \lim_{\epsilon \to 0} \left[\frac{1}{\pi} \int\limits_{-x-\epsilon}^{-x+\epsilon} \frac{c_1 f_1(t) + c_2 f_2(t)}{t-x} dt \right]$$

$$= c_1 \lim_{\epsilon \to 0} \left[\frac{1}{\pi} \int\limits_{-x-\epsilon}^{-x+\epsilon} \frac{f_1(t)}{t-x} dt \right] + c_2 \lim_{\epsilon \to 0} \left[\frac{1}{\pi} \int\limits_{-x-\epsilon}^{-x+\epsilon} \frac{f_2(t)}{t-x} dt \right]$$

$$= c_1 H\{f_1(t)\} + c_2 H\{f_2(t)\}$$

$\square$

Theorem 2 (Derivatives of Hilbert transform). *The Hilbert transform of the derivative of a function $f(t)$ is equal to the derivative of the Hilbert transform of a function i.e. $H\{f'(t)\} = \frac{d}{dx}\hat{f}_H(x)$.*

Proof. By the definition of Hilbert transform, we have

$$\hat{f}_H(x) = \frac{1}{\pi} \fint\limits_{-\infty}^{\infty} \frac{f(t)}{t-x} dt$$

$$= \frac{1}{\pi} \fint\limits_{-\infty}^{\infty} \frac{f(u+x)}{u} du \qquad \left(\text{Substituting } t = u + x \right)$$

Differentiating with respect to x on both sides, we get

$$\frac{d}{dx}\left(\hat{f}_H(x) \right) = \frac{1}{\pi} \fint\limits_{-\infty}^{\infty} \frac{f'(u+x)}{u} du$$

$$= \frac{1}{\pi} \oint_{-\infty}^{\infty} \frac{f'(t)}{t-x} dt \qquad \left(\text{Substituting } u = t - x \right)$$

$$= H\{f'(t)\}$$

□

Theorem 3. *Let* $H\{f(t)\} = \hat{f}_H(x)$ *then* $H\{f(at)\} = \hat{f}_H(ax),\ a > 0.$

Proof. Using the Hilbert transform definition we have,

$$H\{f(at)\} = \frac{1}{\pi} \oint_{-\infty}^{\infty} \frac{f(at)}{t-x} dt$$

$$= \frac{1}{\pi} \oint_{-\infty}^{\infty} \frac{f(au)}{u-ax} du \qquad \left(\text{Substituting } at = u \right)$$

$$= \hat{f}_H(ax)$$

□

Theorem 4. *Let* $H\{f(t)\} = \hat{f}_H(x)$ *then* $H\{f(-at)\} = -\hat{f}_H(-ax),\ a > 0.$

Proof. Using the Hilbert transform definition we have,

$$H\{f(-at)\} = \frac{1}{\pi} \oint_{-\infty}^{\infty} \frac{f(-at)}{t-x} dt$$

$$= -\frac{1}{\pi} \oint_{-\infty}^{\infty} \frac{f(au)}{u+ax} du \qquad \left(\text{Substituting } -at = u \right)$$

$$= -\hat{f}_H(-ax)$$

□

Theorem 5 (Convolution Theorem). *Let* f *and* g *be two real valued functions then*

1. $H(f * g)(x) = (Hf * g)(x) = (f * H(g))(x)$

2. $(f * g)(x) = -(Hf * H(g))(x)$

Proof. 1. Using the definitions of Hilbert transform and Inverse Hilbert transform, we get

$$H(f * g)(x) = \frac{1}{\pi} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dt}{t-x} \left[\int_{-\infty}^{\infty} f(y)g(t-y) dy \right]$$

Substituting $t - y = u$ the above equation reduces to

$$H(f * g)(x) = \frac{1}{\pi} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) dy \left[\frac{g(u) du}{u - (x - y)} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(y)(H(g))(x-y)dy$$

$$= (f * H(g))(x)$$

$$= \frac{1}{\pi}\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} g(y)dy \left[\frac{f(u)du}{u-(x-y)}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} g(y)(H(f))(x-y)dy$$

$$= (Hf * g)(x)$$

2. Replace g as $H(g)$ and $H(H(g)) = -g$ becomes,

$$H(f * g)(x) = \frac{1}{\pi}\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \left[\int\limits_{-\infty}^{\infty} f(y)H(g)(t-y)dy\right]$$

Substituting $(t - y) = u$ the above equation reduces to

$$H(f * g)(x) = \frac{1}{\pi}\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(y)dy \left[\frac{H(g)(u)du}{u-(x-y)}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(y)(-g)(x-y)dy$$

$$= -(H(f) * H(g))(x)$$

$\square$

Theorem 6 (Inversion Theorem). *Suppose $x(t) \in L^p(\mathbb{R})$, $1 < p \leq 2$ then $H\{H\{x(t)\}\} - x(t)$.*

Proof. Using the definition of Hilbert transform twice, we get,

$$F\{H\{H\{x(t)\}\}\} = (-isgn(f))(-isgn(f))f(x(t))$$
$$= i^2(sgn(f))^2 F\{x(t)\}$$
$$= F\{-x(t)\}$$

where F is the Fourier transform operator s. Since $H\{H\{x(t)\}\}$ and $-x(t)$ have the same Fourier transform, we can conclude that $H\{H\{x(t)\}\} = -x(t)$ $\square$

The Hilbert Transform of Strong Analytic Signals

A complex signal whose imaginary part is Hilbert transform of its real part is called analytic signal. The Hilbert transform of a strong analytic signal is given by $z(t) = (f(t) + i\hat{f}(t))$, where $f(t)$ and $\hat{f}(t)$ are real valued functions and $\hat{f}(t) = H\{f(t)\}$ [3, 8].

Then
$$H\{z(t)\} = H\{f(t) + i\hat{f}(t)\}$$
$$= H\{f(t) + iH\{\hat{f}(t)\}$$

$$= \hat{f}(t) - if(t) \qquad \left(\text{Since} H(H(f(t))) = H^{-1}\right)$$
$$= -i(f(t) - i\hat{f}(t))$$
$$= -iz(t) \tag{3}$$

Theorem 7. *The product of $H\{z_1(t)\}z_2(t)$ is identical with the product of $z_1(t)H\{z_2(t)\}$ if $z_1(t)$ and $z_2(t)$ are strong analytic signals.*

Proof. Suppose $z_1(t)$ and $z_2(t)$ are strong analytic signals, then

$$H\{z_1(t)\}z_2(t) = \left(\hat{f}_1(t) - if_1(t)\right)z_2(t) \qquad \left(\text{Using equation (3)}\right)$$
$$= -i\left(f_1(t) + i\hat{f}_1(t)\right)z_2(t)$$
$$= -i\, z_1(t)\, z_2(t)$$
$$= z_1(t)\,(-i)(z_2(t))$$
$$= z_1(t)\left(\hat{f}_2(t) - if_2(t)\right)$$
$$= z_1(t)H\{z_2(t)\} \qquad \left(\text{Using the equation (3)}\right)$$

$\square$

Theorem 8. *The product of $z_1(t)z_2(t)$ is identical with the product of $iH\{z_1(t)\}z_2(t)$ and $iz_1(t)H\{z_2(t)\}$ if $z_1(t)$ and $z_2(t)$ are strong analytic signals.*

Proof. Suppose $z_1(t)$ and $z_2(t)$ are strong analytic signals, then

$$z_1(t)z_2(t) = \left(f_1(t) + i\hat{f}_1(t)\right)z_2(t)$$
$$= i\left(\hat{f}_1(t) - if_1(t)\right)z_2(t)$$
$$= iH\{z_1(t)\}z_2(t) \qquad \left(\text{Using the equation (3)}\right) \tag{4}$$

$$\text{Also } z_1(t)z_2(t) = z_1(t)\left(f_2(t) + i\hat{f}_2(t)\right)$$
$$= z_1(t)i\left(\hat{f}_2(t) - if_2(t)\right)$$
$$= iz_1(t)H\{z_2(t)\} \tag{5}$$

From the equation (4) and (5) we get,

$$iH\{z_1(t)\}z_2(t) = iz_1(t)H\{z_2(t)\}$$

$\square$

3   Examples on Hilbert transforms

Example 1: Find the Hilbert transforms of the following functions:

$$\text{(i) } f(t) = 1 \qquad\qquad \text{(ii) } f(t) = \frac{t}{(t^2 + a^2)}, \ a > 0$$

$$\text{(iii) } f(t) = \sin \omega t \qquad\qquad \text{(iv) } f(t) = \exp(iat), a > 0$$

(i) Using the Cauchy Principal Value definition of the Hilbert transform we have,

$$H\{1\} = \frac{1}{\pi} \lim_{R \to 0} \left[\int_{-\infty}^{x-R} \frac{1 \, dt}{t - x} + \int_{x+R}^{\infty} \frac{1 \, dt}{t - x} \right]$$

$$= \lim_{R \to 0} \left[\log \left\{ \frac{-\left(1 - \dfrac{x}{R}\right)}{\left(1 + \dfrac{x}{R}\right)} \right\} \right]$$

$$= \log(-1)$$

$$= i\pi$$

Considering the real part of the above expression, we have $H\{1\} = 0$.

Thus $H\{c\} = 0$ for any constant c.

(ii) By the definition of the Hilbert transform we have,

$$\hat{f}_H(x) = \frac{1}{\pi} \oint_{-\infty}^{\infty} \frac{t \, dt}{(t^2 + a^2)(t - x)} \tag{6}$$

Let $\dfrac{t}{(t^2 + a^2)(t - x)} = \dfrac{A}{(t - x)} + \dfrac{Bt + C}{(t^2 + a^2)}$

On solving, we get the values of A, B and C as

$$A = \frac{x}{(x^2 + a^2)}, \ B = \frac{-x}{(x^2 + a^2)} \text{ and } C = \frac{a^2}{(x^2 + a^2)}$$

Substituting these values equation (6) becomes

$$\hat{f}_H(x) = \frac{1}{\pi} \frac{1}{(a^2 + x^2)} \left[a^2 \oint_{-\infty}^{\infty} \frac{dt}{(t^2 + a^2)} + x \oint_{-\infty}^{\infty} \frac{dt}{(t - x)} - x \oint_{-\infty}^{\infty} \frac{t \, dt}{(t^2 + a^2)} \right]$$

$$= \frac{a}{(a^2 + x^2)}$$

(iii) By the definition of the Hilbert transform we have,

$$\hat{f}_H(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin\{\omega(t - x) + \omega x\}}{(t - x)} \, dt$$

$$= \frac{\cos \omega x}{\pi} \int_{-\infty}^{\infty} \frac{\sin \omega(t - x)}{(t - x)} \, dt + \frac{\sin \omega x}{\pi} \int_{-\infty}^{\infty} \frac{\cos \omega(t - x)}{(t - x)} \, dt$$

Substituting $t - x = U$ in the above expression we get

$$\hat{f}_H(x) = \frac{\cos \omega x}{\pi} \int\limits_{-\infty}^{\infty} \frac{\sin \omega U}{U} dU + \frac{\sin \omega x}{\pi} \int\limits_{-\infty}^{\infty} \frac{\cos \omega U}{U} dU$$

$$= \cos \omega x$$

(iv) By the definition of the Hilbert transform we have,

$$\hat{f}_H(x) = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \frac{e^{iat}}{t - x} dt$$

$$= \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \frac{\cos at}{(t - x)} dt + \frac{i}{\pi} \int\limits_{-\infty}^{\infty} \frac{\sin at}{(t - x)} dt$$

$$= i \left(\cos ax - \frac{\sin ax}{i} \right)$$

$$= i(e^{iax})$$

Example 2: Find the inverse Hilbert transforms of the following functions:

(i) $f(x) = \dfrac{a}{(a^2 + x^2)}$, $a > 0$ (ii) $f(x) = \cos \omega x$

(iii) $f(x) = i(e^{iax})$, $a > 0$

(i) Using the definition of inverse Hilbert transform, we have

$$H^{-1} \left\{ \frac{a}{a^2 + x^2} \right\} = \frac{-1}{\pi} \oint\limits_{-\infty}^{\infty} \frac{a}{(a^2 + x^2)(x - t)} dx \qquad (7)$$

Let $\dfrac{a}{(a^2 + x^2)(x - t)} = \dfrac{A}{(x - t)} + \dfrac{Bx + C}{(a^2 + x^2)}$

On solving, we get the A, B and C values as

$$A = \frac{a}{(a^2 + t^2)}, \quad B = \frac{-a}{(a^2 + t^2)} \text{ and } C = \frac{-at}{a^2 + t^2}$$

Substituting these equation (7) becomes,

$$H^{-1}\{\hat{f}_H(x)\} = \frac{-1}{\pi} \frac{a}{(a^2 + t^2)} \left[\oint\limits_{-\infty}^{\infty} \frac{dx}{(x - t)} - \int\limits_{-\infty}^{\infty} \frac{x dx}{(a^2 + x^2)} - t \int\limits_{-\infty}^{\infty} \frac{dx}{(a^2 + x^2)} \right]$$

$$= \frac{t}{(a^2 + t^2)}$$

(ii) Using the definition of inverse Hilbert transform, we have

$$H^{-1}\{\hat{f}_H(x)\} = \frac{-1}{\pi} \int\limits_{-\infty}^{\infty} \frac{\cos\{\omega(x-t)+\omega t\}}{(x-t)} dx$$

$$= \frac{-\cos\omega t}{\pi} \int\limits_{-\infty}^{\infty} \frac{\cos\omega(x-t)}{(x-t)} dx - \frac{\sin\omega t}{\pi} \int\limits_{-\infty}^{\infty} \frac{\sin\omega(x-t)}{(x-t)} dx$$

Substituting $x - t = U$ in the above expression becomes,

$$H^{-1}\{\hat{f}_H(x)\} = \frac{-\cos\omega t}{\pi} \int\limits_{-\infty}^{\infty} \frac{\cos\omega U}{U} dU + \frac{\sin\omega t}{\pi} \int\limits_{-\infty}^{\infty} \frac{\sin\omega U}{U} dU$$

$$= \sin\omega t$$

(iii) Using the definition of inverse Hilbert transform, we have

$$H^{-1}\{\hat{f}_H(x)\} = \frac{-1}{\pi} \oint\limits_{-\infty}^{\infty} i\frac{(e^{iax})}{(x-t)} dx$$

$$= \frac{-i}{\pi} \int\limits_{-\infty}^{\infty} \frac{\cos ax}{(x-t)} dx - \frac{i^2}{\pi} \int\limits_{-\infty}^{\infty} \frac{\sin ax}{(x-t)} dx$$

$$= \exp(iat)$$

4   Applications of Hilbert Transforms

Hilbert transform is widely used in signal processing. It has wide variety of application in Fluid mechanics, aerodynamics and electronics. It plays an important role in electro-cardiography, Hilbert-Huang transform is a popular method for spectral analysis for non-linear and non-stationary process and modulation [9, 10].

Applying the Hilbert Transforms to Boundary Value Problems

Solve the Laplace equation

$$u_{xx} + u_{yy} = 0 \text{ where } -\infty < x < \infty \text{ and } y > 0$$

with the boundary conditions

$$u_x(x,y) = f(x) \text{ on } y = 0, \text{ for } -\infty < x < \infty$$
$$u(x,y) \to 0 \text{ as } r = (x^2 + y^2)^{\frac{1}{2}} \to \infty$$

The Fourier transform is defined by

$$F\{f(x)\} = F(s) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{-isx} f(x) dx$$

where F is called Fourier transform operator gives the solution for $U(s, y)$ as

$$U(s, y) = \frac{F(s)}{is} \exp(-|s|y)$$
$$= F(s)G(s)$$

where $G(s) = (is)^{-1} \exp(-|s|y)$

So that $g(x) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{x}{y}\right)$

Using the Convolution theorem the solution is obtained as,

$$u(x, y) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(t)g(x - t)dt$$

$$= \frac{1}{\pi} \int\limits_{-\infty}^{\infty} f(t) \tan^{-1}\left(\frac{x - t}{y}\right)dt \quad (\text{Since } y = 0)$$

$$u_y(x, 0) = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \frac{f(t)}{t - x}dt$$

$$= H\{f(t)\}$$

Thus, the Hilbert of the tangential derivative $u_x(x, 0) = f(x)$ is the normal derivative $u_y(x, 0)$ on the boundary at $y = 0$.

5   Results and Discussions

In this paper we have studied the definition of Hilbert transform and solved a few basic properties and examples on Hilbert transform. We also learnt about analytic signals and how to find the Hilbert transform of an analytic signal. Application of Hilbert transform to Boundary Value Problem was studied with an example.

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