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Study of Fluid Flows in Forced Convection using DTM-Padé Approximation

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Abstract: *The fluid flow equations of a viscous incompressible fluid past a thin semi-infinite flat plate at a constant wall temperature are derived and solutions of these equations are investigated using DTM-Padé approximation. Velocity and temperature profiles are plotted. The velocity profile in the laminar region is approximately parabolic and becomes flatter in turbulent flow. The temperature of the fluid increases with the increasing heat generation parameter and the decreasing Prandtl number. The results obtained using DTM-Padé approximation are compared using numerical solution for validation.*

Keywords: *Forced Convection, Thermal Boundary Layer, DTM-Padé Approximation.*

AMS Subject Classification: *74A05, 76E06, 41A21.*

1 Introduction

A boundary layer is a thin layer of viscous fluid close to the solid surface of a wall in contact with a moving stream in which the flow velocity varies from zero at the wall up to U_∞ at the boundary, which approximately corresponds to the free stream velocity [1]. For heat transfer, the boundary layer is known as thermal boundary layer and its thickness can be defined as the distance at which fluid attains 99% of free stream temperature. A thermal boundary layer is analogous to velocity boundary layer and is formed. When a fluid flows past a heated or cooled bodies the heat is transferred by conduction, convection and radiation.

Heat transfer is complicated since it involves fluid motion as well as heat conduction [2]. The fluid motion enhances heat transfer (the higher the velocity the higher the heat transfer rate) [3]. Convection is the circular motion that happens when warmer air or liquid-which has faster moving molecules, making it less dense-rises, while the cooler air or liquid drops down. Convection is a major factor in atmospheric weather.

Main purpose of convective heat transfer analysis is to determine flow field, temperature field in fluid and heat transfer coefficients. In a forced convection, the Nusselt Number depends on the rate of heat transfer through a boundary layer from a surface hotter or cooler than the air passing over it, a process analogous to the transfer of momentum by skin friction [4]. The Nusselt number is therefore expected to be a function of the Reynolds number modified by the ratio of boundary layer thickness for heat(tH) and for momentum(tM). The ratio tH/tM is a function of the Prandtl Number defined by (ν/κ).

Karvinen [5] presented an approximate method for calculating heat transfer from a flat plate in forced flow and compared the results with experimental data and previous results obtained in [6] for the case of combined convective heat exchange with the environment, conduction in the plate and internal heat sources. Forced convection conjugate heat transfer in a laminar

plane wall jet was considered by Kanna and Das [7]. A problem of conduction-convection in fins [8, 9] and in cavities [5, 8] and the combined effect of conduction and radiation in a T-Y shaped fin [10] are carried out in recent years.

2   Mathematical Formulation

Consider the steady flow of a viscous incompressible fluid past a thin semi-infinite flat plate at a constant temperature T_w placed along the direction of a uniform stream of velocity U_∞ and temperature T_∞ [1]. Let the origin of co-ordinates be at the leading edge of the plate, the $x-$axis along the plate and $y-$axis normal to it. Equations for two dimensional, laminar, steady boundary layer flow are,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + v\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = a\frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho C_p}\left(\frac{\partial u}{\partial y}\right)^2, \tag{3}$$

where $a = \dfrac{k}{\rho C_p}$ (Thermal Diffusivity).

In the present case, $U(x) = U_\infty$ (constant). Thus (2) reduces to,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = v\frac{\partial^2 u}{\partial y^2}, \tag{4}$$

with the boundary conditions,

$$y = 0; u = v = 0, T = T_w \text{ (Isothermal)},$$
$$\frac{\partial T}{\partial y} = 0 \text{ (Adiabatic)}, \tag{5}$$
$$y = \infty; u = U_\infty, T = T_\infty.$$

2.1   Integrals of the Thermal Boundary Layer Equation with Prandtl Number of the Fluid $(\mathrm{Pr} = 1)$

A simple integral of the equation (3) can be obtained immediately if the frictional heat is neglected and the Prandtl number of the fluid is unity. In such cases the two equations are,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{6}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = a\frac{\partial^2 T}{\partial y^2}, \tag{7}$$

with the boundary conditions,

$$y = 0; u = v = 0, T = T_w,$$
$$y = \infty; u = U_\infty, T = T_\infty. \tag{8}$$

When $\nu = a$, we get $\Pr = \dfrac{\nu}{a} = 1$. Thus, equation (7) becomes identical to equation (6) with boundary conditions if $\dfrac{T - T_w}{T_\infty - T_w}$ is replaced by $\dfrac{u}{U_\infty}$. Hence,

$$\frac{T - T_w}{T_\infty - T_w} = \frac{u}{U_\infty}, \text{ (or) } \frac{T - T_\infty}{T_w - T_\infty} = 1 - \frac{u}{U_\infty} (\Pr = 1). \tag{9}$$

It is known as Crocco's first integral. We can also write this as,

$$\frac{\partial}{\partial y}\left(\frac{u}{U_\infty}\right) = \frac{\partial}{\partial y}\left(\frac{T - T_w}{T_\infty - T_w}\right). \tag{10}$$

This shows that the heat-flux and the skin-friction are proportional to each other. To get the exact relationship between the two, we write the value of the local Nusselt number, which is given by,

$$\mathrm{Nu}(x) = -\frac{x\left(\dfrac{\partial T}{\partial y}\right)_o}{T_w - T_\infty} = \frac{x}{U_\infty}\left(\frac{\partial u}{\partial y}\right)_0, \tag{11}$$

$$\mathrm{Nu}(x) = \frac{x}{\mu U_\infty} T_w = \frac{x}{\mu U_\infty}\left(\frac{1}{2}\rho U_\infty^2 C_f\right). \tag{12}$$

Thus,

$$\mathrm{Nu}(x) = \frac{1}{2}\mathrm{Re}_x C_f,$$

where $\mathrm{Re}_x = \dfrac{U_\infty x}{\nu}$ and $\nu = \dfrac{\mu}{\rho}$.

This is known as Reynold's Analogy. If the frictional heat is not neglected but the wall is insulated then another simple integral of equation (3) is possible again when $\Pr = 1$. We have $T = T(u)$ then equation (3) can be written as,

$$u\frac{dT}{du}\frac{\partial u}{\partial x} + v\frac{dT}{du}\frac{\partial u}{\partial y} = a\frac{\partial}{\partial y}\left[\frac{dT}{du}\frac{\partial u}{\partial y}\right] + \frac{\mu}{\rho C_p}\left(\frac{\partial u}{\partial y}\right)^2,$$

$$(\nu - a)T_u\frac{\partial^2 u}{\partial y^2} = \left[aT_{uu} + \frac{\mu}{\rho C_p}\right]\left(\frac{\partial u}{\partial y}\right)^2. \tag{13}$$

Equation (13) will be identically satisfied. Thus $T = T(u)$ will be the solution of equation (3) if,

$$\nu = a \text{ and } T_{uu} = -\frac{\mu}{a\rho C_p},$$

$$\Pr = 1 \text{ and } T_{uu} = -\frac{\nu}{aC_p} \quad \left(\text{since } \nu = \frac{\mu}{\rho}\right),$$

$$T_{uu} = -\frac{1}{C_p} \quad (\text{since } \nu = a),$$

where the constants of integration vanishes, since at $y = 0$, $u = 0$, $\dfrac{\partial u}{\partial y} \neq 0$ and $\dfrac{\partial T}{\partial y} = 0$. That implies $T_u = 0$. Integrating and using the boundary condition at infinity, that is $u = U_\infty$ and $T = T_\infty$, we obtain, $T_u = -\dfrac{u}{C_p}$,

$$[T]_{T_\infty}^T = -\left[\frac{u^2}{2C_p}\right]_{U_\infty}^u,$$

$$T - T_\infty = -\frac{1}{2C_p}[u^2 - U_\infty^2] + \text{constant},$$

$$T - T_\infty = -\frac{u^2}{2C_p} + \frac{U_\infty^2}{2C_p},$$

$$T - T_\infty = \frac{U_\infty^2}{2C_p}\left[1 - \frac{u^2}{U_\infty^2}\right],$$

$$\frac{T - T_\infty}{U_\infty^2/2C_p} = 1 - \left(\frac{u}{U_\infty}\right)^2 \text{ [since Pr} = 1]. \tag{14}$$

This is known as Crocco's second integral.

3 Integrals of the Thermal Boundary Layer Equation for arbitrary values of the Prandtl Number (Pr)

For the solution of equation (3), we shall require the velocity distribution which was obtained by Blasius and is as follows,

$$u = U_\infty \phi'(\eta),$$
$$v = \frac{1}{2}\sqrt{\frac{\nu\, U_\infty}{x}}(n\phi'(\eta) - \phi(\eta)), \tag{15}$$

where $\eta = y\sqrt{\frac{U_\infty}{\nu\, x}}$ is the similarity variable. We have,

$$u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2},$$

$$\frac{U_\infty^2}{x}\phi'''(\eta) + \frac{U_\infty^2}{2x}\phi''(\eta)\phi(\eta) = 0,$$

$$2\phi''' + \phi\phi'' = 0. \tag{16}$$

The function $\phi(\eta)$ satisfies the above differential equation with the boundary conditions,

$$\eta = 0 : \phi = \phi' = 0,$$
$$\eta = \infty : \phi' = 1. \tag{17}$$

In order to obtain a complete integral of equation (3) for an isothermal wall, it will be easier to calculate first the solution of it when the dissipation term is neglected, i.e. the solution of the cooling problem with a prescribed value of $(T_w - T_\infty)$ and then another solution when the frictional heat is accounted but the wall is adiabatic, i.e. the problem of plate thermometer. Since equation (3) is a linear differential equation, to get the complete integral, the two solutions may then be properly superimposed.

4   Solution of the Cooling Problem

Consider equation (7),

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = a\frac{\partial^2 T}{\partial y^2}, \tag{18}$$

with the boundary conditions

$$y = 0: \ T = T_w \text{ and } y = \infty: \ T = T_\infty. \tag{19}$$

We have,

$$\theta_1 = \frac{T - T_\infty}{T_w - T_\infty},$$
$$T - T_\infty = \theta_1(\eta)(T_w - T_\infty), \tag{20}$$
$$T = T_\infty + \theta_1(\eta)(T_w - T_\infty).$$

Solution of equation (18) in which θ_1 is a solution of the similarly variable η only or in other words we look for a similar solution of θ_1.

Differentiate equation (20) w.r.t. x,

$$\frac{\partial T}{\partial x} = (T_w - T_\infty)\theta_1'(\eta)\frac{\partial \eta}{\partial x},$$

$$\frac{\partial T}{\partial x} = (T_w - T_\infty)\theta_1'(\eta)\left[\frac{-y}{2x}\sqrt{\frac{U_\infty}{\nu\, x}}\right] \quad \text{[since from equation (15)].} \tag{21}$$

Differentiate equation (20) w.r.t. y,

$$\frac{\partial T}{\partial y} = (T_w - T_\infty)\theta_1'(\eta)\frac{\partial \eta}{\partial y},$$

$$\frac{\partial T}{\partial y} = (T_w - T_\infty)\theta_1'(\eta)\left[\sqrt{\frac{U_\infty}{\nu\, x}}\right]. \tag{22}$$

Differentiate equation (22) w.r.t. y,

$$\frac{\partial^2 T}{\partial y^2} = (T_w - T_\infty)\theta_1''(\eta)\frac{U_\infty}{\nu\, x}. \tag{23}$$

Substituting the values of equations (15), (21), (22), (23) in equation (18), we get

$$U_\infty\phi'(\eta)(T_w - T_\infty)\theta_1'(\eta)\left(\frac{-y}{2x}\right)\sqrt{\frac{U_\infty}{\nu\, x}} + \frac{1}{2}\sqrt{\frac{\nu\, U_\infty}{x}}\left[\eta\phi'(\eta) - \phi(\eta)\right](T_w - T_\infty)\theta_1'(\eta)\sqrt{\frac{U_\infty}{\nu\, x}}$$
$$= a(T_w - T_\infty)\theta_1''(\eta)\frac{U_\infty}{\nu\, x}.$$

Dividing throughout by $\frac{aU_\infty}{\nu\, x}$,

$$-\phi(\eta)\theta_1'(\eta)\frac{\nu}{2a} = \theta_1''(\eta),$$

$$\theta_1''(\eta) = -\frac{1}{2}\phi(\eta)\theta_1'(\eta)\frac{\nu}{a}.$$

But $\dfrac{\nu}{a} = \mathrm{Pr}$.

$$\theta_1'' + \frac{\mathrm{Pr}}{2}\phi\theta_1' = 0, \tag{24}$$

with the boundary conditions

$$\eta = 0 : \theta_1 = 1, \quad \text{and} \quad \eta = \infty : \theta_{\prime} = 0. \tag{25}$$

From equation (24) we have

$$\theta_1'' + \frac{\mathrm{Pr}}{2}\phi\theta_1' = 0,$$

$$\frac{\theta_1''}{\theta_1} = -\frac{1}{2}\mathrm{Pr}\phi,$$

$$\theta_1'' + \frac{\mathrm{Pr}}{2}\phi\theta_1' = 0,$$

$$\frac{\theta_1''}{\theta_1'} = -\frac{1}{2}\mathrm{Pr}\left[\frac{-2\phi'''}{\phi''}\right] \quad \text{[since from equation (16)]},$$

$$\frac{\theta_1''}{\theta_1'} = \mathrm{Pr}\left[\frac{\phi'''}{\phi''}\right]. \tag{26}$$

Integrating w.r.t. η, we get

$$\frac{\theta_1'}{\theta_1} = \mathrm{Pr}\left[\frac{\phi''}{\phi'}\right].$$

Integrating once again w.r.t. η, we get

$$log\theta_1' = \mathrm{Pr}log\phi'' + logA,$$

where A is the constant of integration.

$$log\theta_1' = log(\phi'')^{\mathrm{Pr}} + logA,$$

$$log\left(\frac{\theta_1'}{A}\right) = log[\phi'']^{\mathrm{Pr}},$$

$$\frac{\theta_1'}{A} = [\phi(\eta)]^{\mathrm{Pr}},$$

$$\theta_1' = A[\phi''(\eta)]^{\mathrm{Pr}}. \tag{27}$$

Integrating equation (27) w.r.t. η from ∞ to η, we get

$$\theta_1(\eta, \mathrm{Pr}) = A\int_{\infty}^{\eta}[\phi''(\eta)]^{(\mathrm{Pr})}d\eta,$$

$$A = \frac{-\theta_1(\eta, \mathrm{Pr})}{\int_{\eta}^{\infty}[\phi''(\eta)]^{\mathrm{Pr}}d\eta}. \tag{28}$$

Using the first boundary condition of equation (25), i.e. when $\eta = 0$, $\theta_1 = 1$,

$$A = \frac{-1}{\int_{0}^{\infty}[\phi''(\eta)]^{\mathrm{Pr}}d\eta}, \tag{29}$$

Equating equations (28) and (29), we get

$$\frac{-\theta_{\prime}(\eta, \mathrm{Pr})}{\int_{\eta}^{\infty}[\phi''(\eta)]^{\mathrm{Pr}}d\eta} = \frac{-1}{\int_{0}^{\infty}[\phi''(\eta)]^{\mathrm{Pr}}d\eta},$$

$$\theta_1(\eta, \mathrm{Pr}) = \frac{\int_{\eta}^{\infty}[\phi''(\eta)]^{\mathrm{Pr}}d\eta}{\int_{0}^{\infty}[\phi''(\eta)]^{\mathrm{Pr}}d\eta}.$$

When the Prandtl number (Pr) of the fluid is unity, the above equation becomes

$$\theta_1 = \frac{[\phi'(\eta)]_{\eta}^{\infty}}{[\phi'(\eta)]_{0}^{\infty}}.$$

5   Results and Discussions

The velocity profiles for Pr=1 and temperature profile for varying Prandtl number is plotted in graphs in Figures 1 and 2 respectively.The velocity profile in the laminar region is approximately parabolic and becomes flatter in turbulent flow. The temperature of the fluid increases with the increasing heat generation parameter and the decreasing Prandtl number. In graphs in Figure 3 results obtained using DTM-Pad$é$ approximation are validated using Runge-Kutta Felhberg Method.
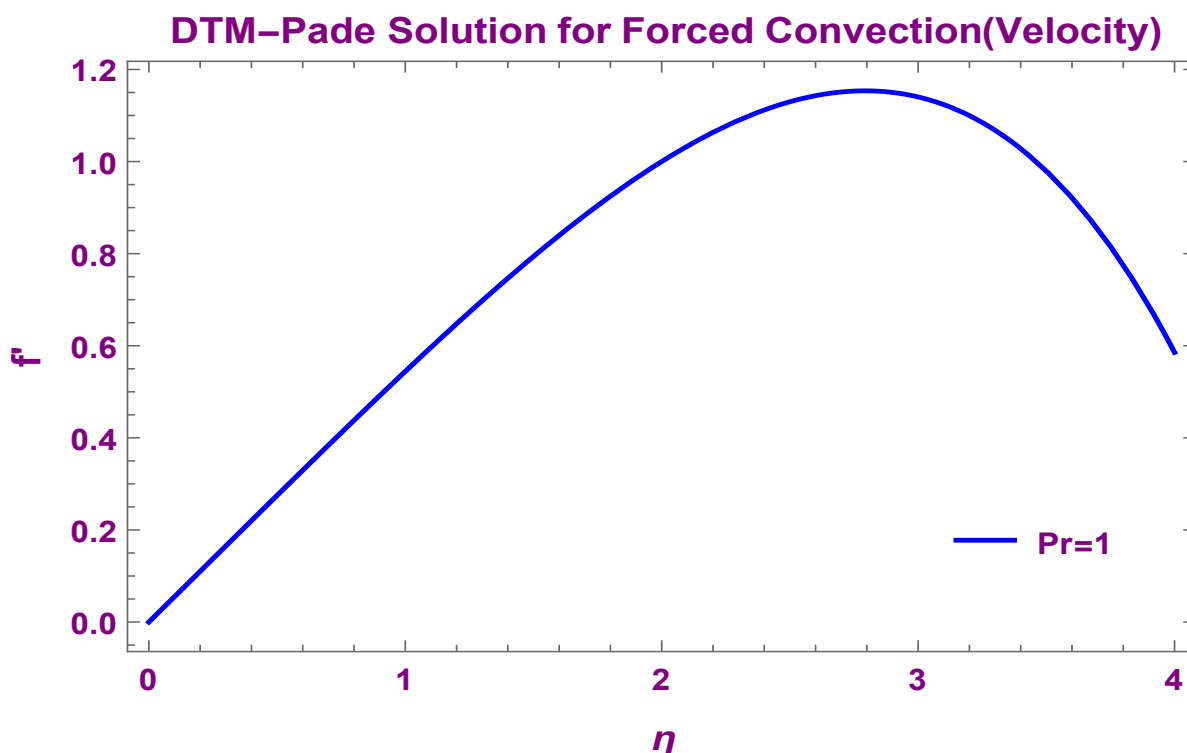
Figure 1: Velocity profile in the laminar boundary layer with $\mathrm{Pr} = 1$
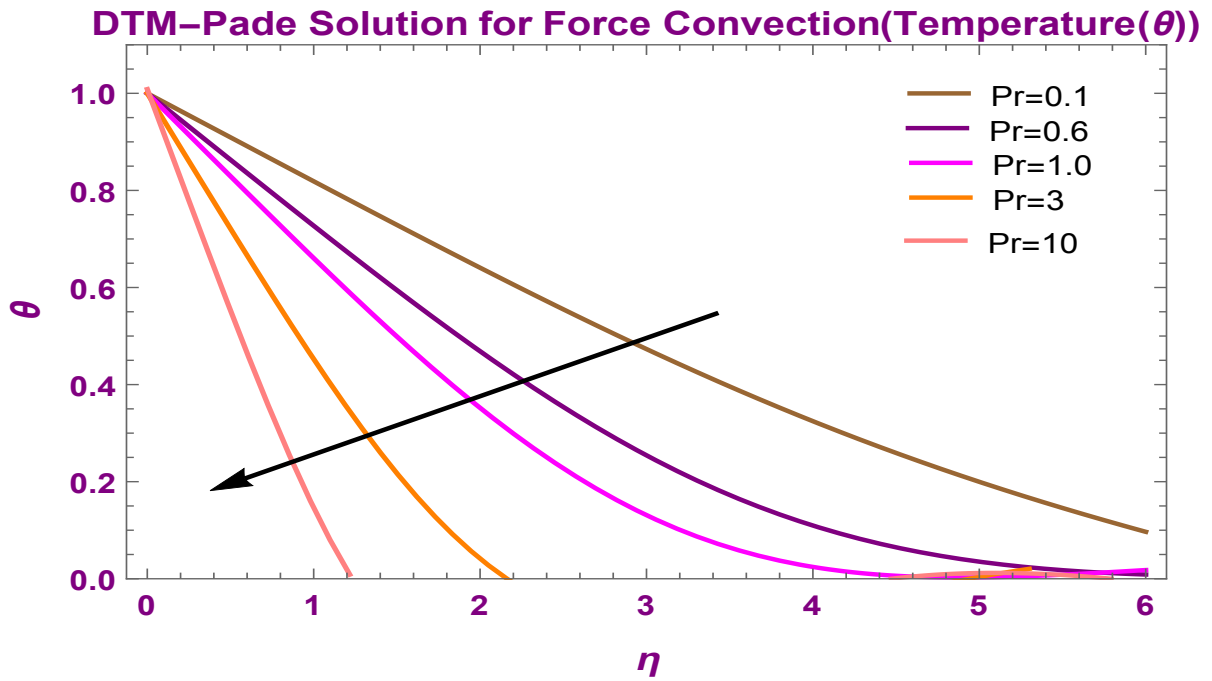
Figure 2: Temperature distribution in the laminar boundary Layer for different Prandtl numbers
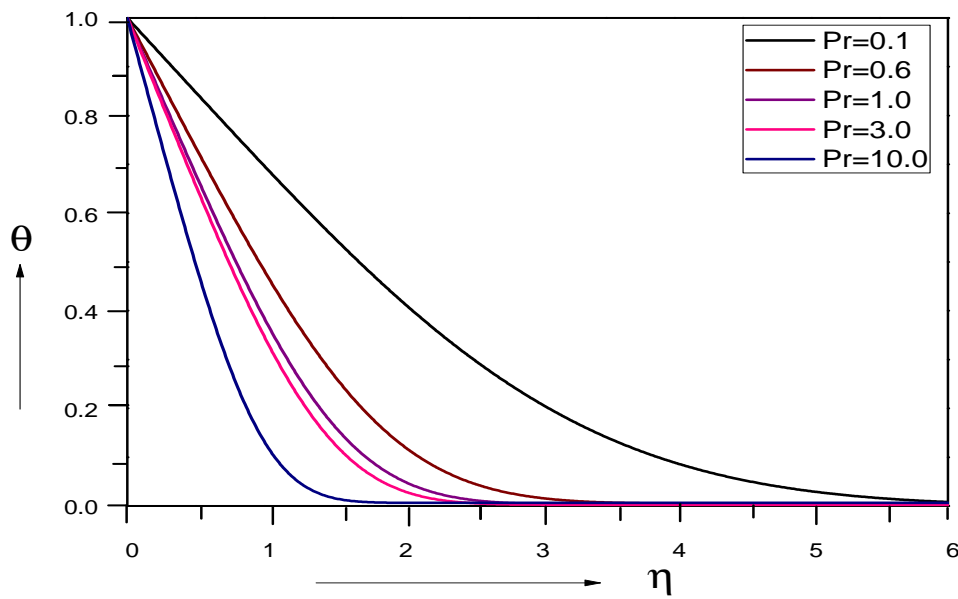
Figure 3: Numerical solution using Runge-Kutta-Fehlberg method

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Different forms of Euler Equations, Expansion Fan and Prandtl Meyer Function

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Abstract: *In this paper we learn about the governing equations of gas dynamics as expressions of conservation that requires three fundamental quantities- mass, momentum and energy; and also we expressed the Euler equation in differential form. Furthermore, we will study about the second law of thermodynamics, equations of state for a perfect gas and also derived the Prandtl-Meyer function. We also showed the Euler equations in case of isothermal flow, also proved the conservation of mass and momentum for any arbitrary control volume $[a, b]$ in steady one-dimensional gas dynamics.*

Keywords: *Euler equation, Boyle's law, Charles' law, Expansion Fan, Prandtl Meyer function.*

AMS Subject Classification: *76N15.*

1   Introduction

We learnt that our entire universe is made up of matter which is anything that has mass and occupies space and this matter exists in three states- solids, liquids and gases. These classifications are made on the basis of the arrangement of molecules in a system. Gases plays an important role in our daily life when compared to solids and liquids because we can live without food and water for some days but we can't live without oxygen for few minutes, like this there are so many reasons which leads to work on properties of gases and its applications extends the effective use of gases in different fields, so we learn more about different types of gases and how they are helpful for us.

Uses of gases in our daily life:
- Oxygen-for respiration purpose and in welding works.
- Carbon dioxide - used as anti-explosive in combination with other gases.
- Nitrogen-to create oxygen free environment.
- Acetylene - for gas welding.
- Natural gas-for cooking.
- Neon, Argon ,xenon in electric bulbs and tube lights.
- Sulfur dioxide for preservation of fruits and wine making.

1.1   Gas laws

In general gas laws are used to analyzes the behavior of gases when slight changes in physical conditions like pressure, temperature or volume occurs and also gives the relation between them. Now let us study some important gas laws.

1. **Boyle's law:** In 1662 Robert Boyle introduced this law, It states that under a constant temperature, pressure (P) of the gas is inversely proportional to its volume (V), $V \propto \dfrac{1}{P}$.

2. **Charles' law:** In 1787 Jacques Charles introduced this law, It states that under a constant pressure, temperature (T) the gas is directly proportional to its volume (V), $V \propto T$.

3. **Gay-Lussac's law:** In 1808 Joseph Gay-Lussac introduced this law it states that under a constant volume, pressure of the gas is directly proportional to its temperature (T), $P \propto T$.

4. **Avogadro's law:** In 1811 Amedeo avogadro combined the conclusions of Dalton's atomic theory and Gay Lussac's law to give another important gas law called the Avogadro's law this law states that at constant temperature and pressure, volume of gas constitutes an equal number of molecules of that gas, $V \propto n$. The number of molecules present in a mole of any gas is called as Avagadro's constant and its value is $6.022 * 10^{23}$.

2 The differential form of Euler equation

2.1 Conservation of mass
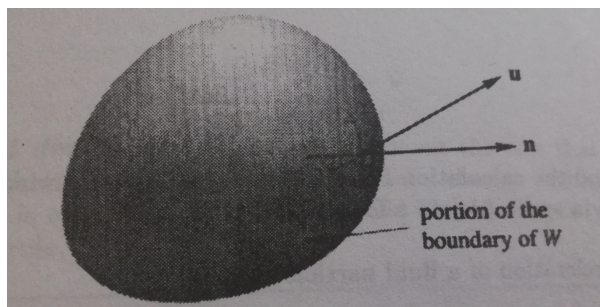
Figure 1

Let W be a fixed sub region of D (W is independent of time). The rate of change of mass in W is given by,

$$\frac{d}{dt}m(W,t) = \frac{d}{dt}\int_W \rho(x,t)dV = \int_W \frac{\partial \rho}{\partial t}(x,t)dV. \tag{1}$$

Let the boundary of W be ∂W, which is assumed to be smooth. Let n be the unit outward normal defined at points of ∂W and dA be the area element on ∂W. Where $\rho \cdot n$ denote the mass flow rate per unit area and $u \cdot n$ be the volume flow rate across ∂W per unit area.

In figure 1, the mass which crossing the boundary ∂W per unit time is equal to the surface integral of $\rho u \cdot n$ over ∂W. We can exactly state the principle of conservation of momentum as: The rate of increase of mass in W equal to the rate at which mass crossing ∂W in the inward direction that is,

$$\frac{d}{dt}\int_W \rho \, dV = - \int_{\partial W} \rho u \cdot n \, dA. \tag{2}$$

It is an integral form of the law of conservation of mass. Using divergence theorem, this statement can also be written mathematically as,

$$\int_W \left[\frac{\partial \rho}{\partial t} + div(\rho u) \right] dV = 0. \tag{3}$$

Since it holds good for all W, and it is equivalent to,

$$\frac{\partial \rho}{\partial t} + div(\rho u) = 0. \tag{4}$$

Equation (4) is the differential form of the law of conservation of mass, and it is also known as the Continuity equation [1].

2.2   Balance of momentum
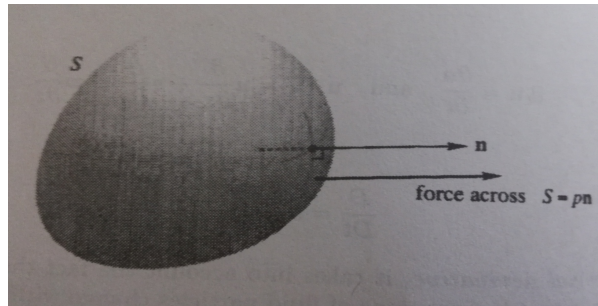
Figure 2

Let the path followed by a fluid particle be $x(t) = (x(t), y(t), z(t))$ then the velocity field is given by,

$$u(x(t), y(t), z(t), t) = (x(t)\dot{x}, y(t)\dot{y}, z(t)\dot{z}). \tag{5}$$

In notation form equation (5) can be written as,

$$u(\pmb{x(t)},t) = \frac{d\pmb{x}}{dt}(t). \tag{6}$$

Equation (6) and the following calculations use standard Euclidean co-ordinates in space and ignoring z for plane flow. Acceleration is given by,

$$\pmb{a}(t) = \frac{d^2}{dt^2}\pmb{x}(t) = \frac{d}{dt}\pmb{u}(x(t), y(t), z(t), t). \tag{7}$$

Using chain rule equation (7) can be written as,

$$\pmb{u}_z = \frac{\partial \pmb{u}}{\partial x}\dot{x} + \frac{\partial \pmb{u}}{\partial y}\dot{y} + \frac{\partial \pmb{u}}{\partial z}\dot{z} + \frac{\partial \pmb{u}}{\partial t}. \tag{8}$$

Using notation we can write the equation (8) as,

$$\pmb{a}(t) = \frac{\partial \pmb{u}}{\partial x}, \quad \pmb{u}_y = \frac{\partial \pmb{u}}{\partial y}, \quad \pmb{u}_z = \frac{\partial \pmb{u}}{\partial z}, \quad \pmb{u}_t = \frac{\partial \pmb{u}}{\partial t}, \tag{9}$$

and

$$\pmb{u}(x, y, z, t) = (u(x, y, z, t), v(x, y, z, t), w(x, y, z, t)). \tag{10}$$

Then we obtain,

$$\pmb{a}(t) = u\pmb{u}_x + v\pmb{u}_y + w\pmb{u}_z + \pmb{u}_t, \tag{11}$$

which can also be expressed as,

$$\pmb{a}(t) = \partial_t \pmb{u} + \pmb{u} \cdot \nabla \pmb{u}, \tag{12}$$

where

$$\partial_t \boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial t} \quad \text{and} \quad \boldsymbol{u} \cdot \nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}. \tag{13}$$

We know that,

$$\frac{D}{Dt} = \partial_t + \mathbf{u} \cdot \nabla. \tag{14}$$

The material derivative that takes into account that the fluid is moving and that the positions of fluid particle change with time.If $f(x, y, z, t)$ is any function of position and time may be scalar or vector, then by chain rule,

$$\frac{d}{dt}f(x(t), y(t), z(t), t) = \partial_t f + \boldsymbol{u}.\nabla f = \frac{Df}{Dt}(x(t), y(t), z(t), t). \tag{15}$$

For every continuum, there are two types of forces which are acting on a piece of material. First,there are forces of stress, they acts on the piece material by forces across its surface by the rest of the continuum. Second, there are external or body forces such as gravity or a magnetic field, which exert a force per unit volume on the continuum. Later, we can examine the stresses more generally, let us define an ideal fluid as one with the following property that for any motion of the fluid there is a function $p(\mathbf{x}, t)$ which is the Pressure such that if surface S in the fluid with a chosen unit normal n, the force of stress exerted across the surface S per unit area at position $\mathbf{x} \in S$ at time t is $p(\mathbf{x}, t)$ $\mathbf{n}$, that is given by,

Force across the surface S per unit area $= p(\mathbf{x}, t)\mathbf{n}$.

We observed that the force is in the direction $\mathbf{n}$ and that the force acts orthogonal to the surface S, it means there are no tangential forces exists(see Figure 2). The physical relevance of the notion (or mathematical theorems we deduce from it) must be checked by experiment. As we shall see later, in ideal fluids many interesting real physical phenomenon are excluded. It is clearly observed that the absence of tangential forces leading that there is no way for rotation to start in a fluid, nor, if it is there at the beginning, to stop.However, here we can detect physical trouble for ideal fluids because of the abundance of rotation in real fluids (near the oars of a rowboat, in tornadoes, etc). If the region in the fluid at a particular instant of time t is W, then the total force exerted on the fluid inside W by means of stress on its boundary is given by,

$$S_{\partial W} = \text{force on } W = -\int_{\partial W} p\mathbf{n} \, dA, \tag{16}$$

Here negative sign indicates that $\mathbf{n}$ points outward. If $\mathbf{e}$ be any fixed vector in space, then divergence theorem gives,

$$e \cdot S_{\partial W} = -\int_{\partial W} pe \cdot \mathbf{n}dA = -\int_{W} div(pe)dV = -\int_{W} (grad\ p) \cdot e \, dV. \tag{17}$$

Thus the total body force is,

$$B = \int_{w} (\rho b) \, dV. \tag{18}$$

Thus on any piece of fluid material, Force per unit volume $= -grad\ p + \rho b$.
By Newton's second law ($f = m \times a$) this lead to the differential form of the balance of momentum,

$$\rho\frac{D\mathbf{u}}{Dt} = -grad\ p + \rho\ b\ [1]. \tag{19}$$

2.3   Conservation of Energy

For three-dimensional space conservation of mass and momentum equation involves four equations or n dimensional space involves $n+1$ equations for ,because the equation for $D\mathbf{u}/Dt$ is a vector equation composed of three scalar equations. However we have five functions: u, ρ and p Thus one might suspect that to specify the fluid motion completely and another one equation is needed. This is in fact true,and conservation of energy will supply the necessary equation in fluid mechanics. This is more complicated situation for general continuum, and issues of general thermodynamics would need to be discussed for a complete treatment. We shall confine ourselves to two special cases here, and later we shall treat another case for an ideal gas. For fluid a moving with velocity field u in a domain D, the kinetic energy contained in a region $W \in D$ is,

$$E_{kinetic} = \frac{1}{2} \int_w \rho \|\mathbf{u}\|^2 dV, \tag{20}$$

where $\|u\|^2 = (u^2 + v^2 + w^2)$ is the square length of the vector function u. We assume that the total energy of the fluid can be written as,

$$E_{total} = E_{kinetic} + E_{internal}, \tag{21}$$

where $E_{internal}$ is the internal energy, which we cannot see on a macroscopic scale and derives from the sources such as inter-molecular vibrations and inter molecular potentials. If energy is pumped into the fluid or if we allow the fluid to do work, results change in E_{total}. Using the transport theorem the rate of change of kinetic energy of a moving portion W_t of fluid is calculated as follows,

$$\frac{d}{dt}E_{kinetic} = \frac{d}{dt}\left[\frac{1}{2}\int_{W_t}\rho\|u\|^2\right] dV,$$

$$= \frac{1}{2}\int_{W_t}\rho\frac{D\|u^2}{Dt}dV,$$

$$= \int_{W_t}\rho\left(u\cdot\left(\frac{\partial u}{\partial t} + (u\cdot\nabla)u\right)\right) dV. \tag{22}$$

Here we use Euclidean coordinates in the following calculation,

$$\frac{1}{2}\frac{D}{Dt}\|u\|^2 = \frac{1}{2}\frac{\partial}{\partial t}(u^2 + v^2 + w^2) + \frac{1}{2}u\frac{\partial}{\partial x}(u^2 + v^2 + w^2)$$

$$+ \frac{1}{2}v\frac{\partial}{\partial y}(u^2 + v^2 + w^2) + \frac{1}{2}w\frac{\partial}{\partial z}(u^2 + v^2 + w^2),$$

$$= u\frac{\partial u}{\partial t} + v\frac{\partial v}{\partial t} + w\frac{\partial w}{\partial t} + u\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x}\right)$$

$$+ v\left(u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} + w\frac{\partial w}{\partial y}\right) + w\left(u\frac{\partial u}{\partial z} + v\frac{\partial v}{\partial z} + w\frac{\partial w}{\partial z}\right),$$

$$\frac{1}{2}\frac{D}{Dt}\|u\|^2 = \mathbf{u} \cdot \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot (\mathbf{u} \cdot \nabla)\mathbf{u}. \tag{23}$$

This is the differential form of the law of conservation of energy [1].

3   Equations of state for a perfect gas

In thermodynamics, macroscopic properties of a system are described by mechanical properties such as kinetic energy and velocity while the microscopic properties of that system described by the thermodynamic properties such as internal energy and enthalpy. Some other properties such as density and pressure are considered either mechanical or thermodynamic properties. By using three thermodynamic properties such as volume, mass, and internal energy, thermodynamic state is defined while the other properties of thermodynamics can be expressed as function of these three is called as equation of state. These are obtained as a result of conservation on the microscopic levels, there are three equations of state which determines all the other equation of state. Assuming that due to direct collision the large number of gas molecules get interacts. On the microscopic level conservation of momentum yields the following equation of state is known as Ideal gas law.

$$p = \rho RT, \tag{24}$$

where R is the gas constant. Typical value R for a sea level air is given by,

$$R = 287 N{\cdot}m/kg{\cdot}K.$$

When a fluid satisfies the thermal equation of state then it is said to be thermally perfect. Again we are assuming that only upon direct collision the large number of gas molecules interacts. Conservation of energy yields the following Caloric equation of state on the microscopic level,

$$e = c_v T, \tag{25}$$

where the specific heat of gas at constant volume is denoted by c_v, for this an equivalent expression can be written as,

$$h = c_p T, \tag{26}$$

where specific heat of gas at constant pressure is denoted by c_p. The specific heats are assumed to be constant and this constant is different for different gases, we have it for sea-level air as $c_v = 717 Nm/kg \cdot K$ and $c_p = 1,004 Nm/kg \cdot K$. The fluid satisfying the equations (24), (25) and (26) is called a perfect gas and gases which not satisfies are called as real gases. The ratio of specific heat is given by,

$$\gamma = \frac{c_p}{c_v}. \tag{27}$$

The equation which relates gas constant and specific heat is given by,

$$c_v = R - c_p, \tag{28}$$

(or)

$$c_p = \frac{\gamma R}{(\gamma - 1)}, \tag{29}$$

$$c_v = \frac{R}{(\gamma - 1)}. \tag{30}$$

Combination of equations (25), (26), (29) and (30) yields the equation of state.

$$P = (\gamma - 1)\rho e = (\gamma - 1)\left(\rho e_T - \frac{1}{2}\rho u^2\right), \tag{31}$$

$$P = (\gamma - 1)\rho e = \gamma \rho e_T - \gamma \frac{1}{2}\rho u^2) - \rho e_T + \frac{1}{2}\rho u^2. \tag{32}$$

4   Entropy and the Second law of Thermodynamics

Entropy: In 1850, Rudolf Clausius introduced this concept, It is the measure of system's thermal energy per unit temperature which is not available for useful work. This entropy helps in finding the measure of the probability of disorder of a macroscopic system. Using Boltzmann's relation entropy is given by,

$$S = k \, ln \, W. \tag{33}$$

where $k = 1.38 \times 10^{-23}$ is Boltzmann's constant and W be the number of microscopic states.

Second law of thermodynamics: Second law of thermodynamics is the most fundamental law in nature. In terms of entropy second law of thermodynamics can be stated as follows: The total entropy of the universe never decreases. For a perfect gas the equation of state giving specific entropy S as a function of specific internal energy and density is given by,

$$S = c_v \, ln \, e - R \, ln \, \rho + \text{constant}. \tag{34}$$

Using perfect gas relation we can rewrite equation (30) as,

$$S = c_v \, ln \, p - c_p \, ln\rho + \text{constant}. \tag{35}$$

Suppose the flow is homentropic, then equate equations (34) and (35) then we get,

$$p = (\text{constant})\rho^\gamma, \tag{36}$$

$$p = (\text{constant})\rho^{\gamma-1}, \tag{37}$$

$$p = (\text{constant})\rho^{(\gamma-1)/2}. \tag{38}$$

For one dimensional flow, second law of thermodynamics can be stated as "Change in total entropy in $[a, b]$ in interval $[t_1, t_2] \geq$ net entropy passing through boundaries of $[a, b]$ in time interval $[t_1, t_2]$". It can be mathematically expressed as follows,

$$\int_a^b [\rho(x,t_2)s(x,t_2) - \rho(x,t_1)s(x,t_1)]dx \geq -\int_{t_1}^{t_2} [\rho(b,t)u(b,t)s(b,t) - \rho(a,t)u(a,t)s(a,t)]dt, \tag{39}$$

where $\rho(x,t)s(x,t)$ is the entropy per unit volume, $s(x,t)$ is entropy per unit mass and $\rho(x,t)s(x,t)u(x,t)$ is the instantaneous entropy flux. Then,

$$S = \int_a^b \rho(x,t)s(x,t)dx. \tag{40}$$

Equation (40) is the total entropy in $[a, b]$ at time t and

$$S = \int_{t_1}^{t_2} \rho(x,t)u(x,t)s(x,t)dt. \tag{41}$$

Equation (41) is the total entropy following past x in the time interval $[t_1, t_2]$, and is known as the total entropy flux.

5   Different forms of Euler equations

Case 1: Derive the Euler equation for isothermal flow,

- $\dfrac{\partial}{\partial t}\rho + \dfrac{\partial}{\partial x}(\rho u) = 0,$

- $\dfrac{\partial}{\partial t}(\rho u) + \dfrac{\partial}{\partial x}(\rho(u^2 + a^2)) = 0,$

where $a^2 = \left(\dfrac{dp}{d\rho}\right) T = RT = \text{constant}.$

Solution: Consider a closed surface A which encloses the volume V entirely which is occupied by the fluid. At position X the density of the fluid will be ρ at the time t, the mass of the fluid enclosed by the surface at any instant is $\int \rho dV$ and the net rate at which mass is flowing outwards across the surface is $\int \rho u \cdot n dA$, where δA and δV are elements of the enclosed volume and the area of the surrounding surface and n be the unit outward normal then mass is conserved in the absence of sources of fluid so we have,

$$\frac{d}{dt}\int \rho dV = -\rho u \cdot n dA. \tag{42}$$

By Divergence theorem,

$$\int \left\{ \frac{\partial \rho}{\partial t} + \nabla \cdot \rho u \right\} dV = 0. \tag{43}$$

Equation (29) holds good for all volumes hence,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho u = 0. \tag{44}$$

Euler equation of motion is given by,

$$\frac{D\vec{q}}{Dt} = F - \frac{1}{\rho}\nabla P. \tag{45}$$

External force F acting on the motion is neglected then,

$$\frac{D\vec{q}}{Dt} = -\frac{1}{\rho}\nabla P, \tag{46}$$

$$\rho\frac{D\vec{q}}{Dt} = -\nabla P. \tag{47}$$

$\vec{q} = (u).$
We know that,

$$\frac{D\vec{u}}{Dt} = \frac{\partial u}{\partial t} + (u \cdot \nabla)u. \tag{48}$$

Equation (48) becomes,

$$\rho\left(\frac{\partial u}{\partial t} + (u \cdot \nabla)u\right) = -\nabla P,$$

$$\rho\left(\frac{\partial u}{\partial t} + (u \cdot \nabla)u\right) = -\frac{\partial p}{\partial x},$$

$$\rho \left(\frac{\partial u}{\partial t} + (u \cdot \nabla)u \right) = -\frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial x}.$$

Using $\frac{\partial p}{\partial \rho} = a^2$ we obtain,

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(u^2 \rho) + a^2\frac{\partial \rho}{\partial x} = u\left(\frac{\partial \rho}{\partial t} + \rho\frac{\partial \rho}{\partial x}\right) + u^2\frac{\partial \rho}{\partial x}. \tag{49}$$

We know the that continuity equation is,

$$\frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} = 0. \tag{50}$$

Substitute equation (50) in equation (49) then we have,

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}\left(\rho(u^2 + a^2)\right) = 0. \tag{51}$$

Case 2: Write the isothermal Euler equation in the form of $\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial t} = 0$ [2].

Solution: Euler equations for an isothermal flow is given by,

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}\left(\rho(u^2 + a^2)\right) = 0. \tag{52}$$

where, $a^2 = \left(\frac{dp}{d\rho}\right)T = RT = $ constant.
We know that,

$$\frac{\partial}{\partial t}(\rho) + \frac{\partial}{\partial t}(\rho u) = 0. \tag{53}$$

From equations (52) and (53) we get,

$$\frac{\partial u}{\partial t} + \frac{1}{\rho}\frac{\partial}{\partial x}\left(\rho(u^2 + a^2)\right) - \frac{u}{\rho}\frac{\partial(\rho u)}{\partial x} = 0, \tag{54}$$

$$\frac{1}{\rho}\frac{\partial}{\partial x}\left(\rho(u^2 + a^2)\right) = \frac{\partial}{\partial x}\left(\frac{1}{\rho} \cdot \rho(u^2 + a^2)\right) - \rho(u^2 + a^2)\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right)$$
$$= \frac{\partial u^2}{\partial x} + \left(\frac{u^2}{\rho}\right)\frac{\partial \rho}{\partial x} + \left(\frac{a^2}{\rho}\right)\frac{\partial \rho}{\partial x}, \tag{55}$$

$$\frac{u}{\rho}\frac{\partial(\rho u)}{\partial x} = \frac{\partial}{\partial x}\left(\frac{u}{\rho} \cdot \rho u\right) - \rho u\frac{\partial}{\partial x}\left(\frac{u}{\rho}\right)$$
$$= \frac{\partial u^2}{\partial x} + \left(\frac{u^2}{\rho}\right)\frac{\partial \rho}{\partial x} - u\frac{\partial u}{\partial x}. \tag{56}$$

Using equations (55) and (56) in equation (54) we get,

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \left(\frac{u^2}{\rho}\right)\frac{\partial \rho}{\partial x} + \left(\frac{a^2}{\rho}\right)\frac{\partial \rho}{\partial x} - \frac{\partial u^2}{\partial x} - \left(\frac{u^2}{\rho}\right)\frac{\partial \rho}{\partial x} + u\frac{\partial u}{\partial x} = 0,$$

$$\frac{\partial u}{\partial t} + \frac{a^2}{\rho}\frac{\partial \rho}{\partial x} + u\frac{\partial u}{\partial x} = 0,$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{a^2}{\rho}\cdot\rho\right) + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0,$$

$$\frac{\partial u}{\partial t} + \frac{\partial a^2}{\partial x} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0,$$

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0. \quad [f(u) = \frac{u^2}{2}].$$

Case 3: For any arbitrary control volume $[a, b]$ in one-dimensional gas dynamics. Prove that,

- $\rho_a u_a = \rho_b u_b$,

- $p_a + \rho_a u_a^2 = p_b + \rho_b u_b^2$.

Solution:
(i) Conservation of mass for an one dimensional flow is stated as: Change in total mass in $[a, b]$ in time interval $[t_1, t_2]$ = net mass passing through the boundaries of $[a, b]$ in time interval $[t_1, t_2]$. This statement can be mathematically expressed as,

$$\int_a^b [\rho(x, t_2) - \rho(x, t_1)]dx = -\int_{t_1}^{t_2} [\rho(b, t)u(b, t) - \rho(a, t)u(a, t)]dt, \qquad (57)$$

where $\rho(x, t)$ is the mass per unit volume, $u(x, t)$ is the velocity and $\rho(x, t)u(x, t)$ is the time rate of flow at point x then $\int_{t_1}^{t_2} \rho(x, t)u(x, t)dt$ is the total mass flux through the surface x. By the definition total mass entering through the point a is equal to the total mass passing through the point b, Therefore the initial and final mass are equal. From equation (57) we get,

$$\int_{t_1}^{t_2} [\rho(b, t)u(b, t) - \rho(a, t)u(a, t)]dt = 0,$$

$$\int_{t_1}^{t_2} \rho(b, t)u(b, t)dt = \int_{t_1}^{t_2} \rho(a, t)u(a, t)dt. \qquad (58)$$

Integrals in equation (58) are equal, then integrands are also equal.

$$\rho(b, t)u(b, t) = \rho(a, t)u(a, t). \qquad (59)$$

Here velocity and density are independent of time. Hence we can write equation (59) in notation form as,

$$\rho_a u_a = \rho_b u_b. \qquad (60)$$

(ii) Conservation of momentum for an one dimensional flow is stated as, "Change in total momentum in $[a, b]$ in time interval $[t_1, t_2]$ = net momentum flow through boundaries of $[a, b]$ in time interval $[t_1, t_2]$+ net momentum change due to pressure on boundaries of $[a, b]$". This statement can be expressed mathematically as follows,

$$\int_a^b [\rho(x, t_2)u(x, t_2) - \rho(x, t_1)u(x, t_1)]dx = -\int_{t_1}^{t_2} [\rho(b, t)u^2(b, t) - \rho(b, t)u^2(b, t)]dt,$$

$$= -\int_{t_1}^{t_2} [p(b,t) - p(a,t)]dt. \tag{61}$$

where $\rho(x,t)u(x,t)$ is momentum per unit volume then $\int_a^b \rho(x,t)u(x,t)dx$ is the total momentum in $[a,b]$ and $\int_{t_1}^{t_2} p(x,t)dt$ is the total momentum change at x due to pressure in time interval $[t_1, t_2]$. By the definition the momentum of flow at point a is equal to the momentum at b. Hence the initial and final momentum are same. Therefore from equation (61) we can write,

$$-\int_{t_1}^{t_2} [\rho(b,t)u^2(b,t) - \rho(b,t)u^2(b,t)]dt - \int_{t_1}^{t_2} [p(b,t) - p(a,t)]dt = 0,$$

$$-\int_{t_1}^{t_2} \rho(b,t)u^2(b,t)dt + \int_{t_1}^{t_2} \rho(b,t)u^2(b,t)dt = \int_{t_1}^{t_2} p(b,t)dt + \int_{t_1}^{t_2} p(a,t)dt,$$

$$\int_{t_1}^{t_2} p(a,t)dt + \int_{t_1}^{t_2} \rho(b,t)u^2(b,t)dt = \int_{t_1}^{t_2} p(b,t)dt + \int_{t_1}^{t_2} \rho(b,t)u^2(b,t)dt,$$

$$\int_{t_1}^{t_2} [p(a,t) + \rho(b,t)u^2(b,t)]dt = \int_{t_1}^{t_2} [p(b,t) + \rho.(b,t)u^2(b,t)]dt. \tag{62}$$

Since the integrals in equation (62) are equal then the integrands of those integrals are also equal. Hence we can write,

$$p(a,t) + \rho(b,t)u^2(b,t) = p(b,t) + \rho(b,t)u^2(b,t). \tag{63}$$

It is observed that momentum is independent of time then equation (63) can be expressed in notation as,

$$p_a + \rho_a u_a^2 = p_b + \rho_b u_b^2. \tag{64}$$

6   Expansion fan

When supersonic flow starts flowing towards itself then it undergoes the compression through the shock and expansion of supersonic flow takes place when it flow out of itself. Both the process are takes place through infinite expansion waves smoothly, this complete process is called as expansion fan. This expansion fan consists of infinite number of Mach waves, where every wave plays a prominent role in resulting the infinitesimal amount of deflection. Typical expansion fan is shown in Figure 3, where supersonic flow turns outward by an angle of deflection θ.

Consider p_1, T_1 and M_1 be the properties of the flow for upstream of the expansion fan or flow before expansion, where p_2, T_2 and M_2 for downstream of the expansion fan or flow after expansion due to outward deflection angle θ. Since the process of expansion is continuous and smooth between infinite Mach waves. Let us consider a wave across the upstream of velocity V and Mach number M. Angle made by this Mach wave with upstream velocity vector is μ and

dV be the velocity change brought by the Mach wave by turning through an angle $d\theta$. Then $V + dV$ and $M + dM$ are denoting the downstream velocity and downstream Mach number respectively. We can use $\sin$ law for the following velocity triangle as show in Figure 4.
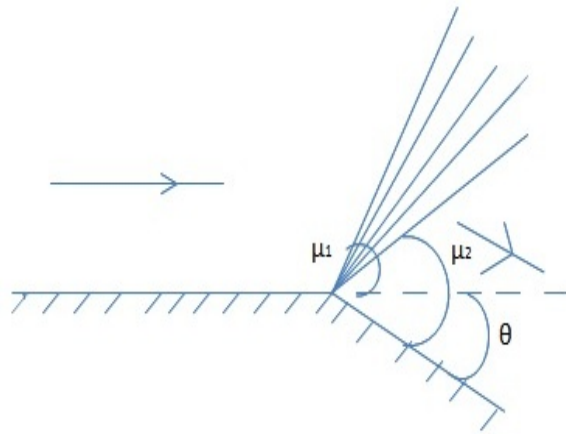
Figure 3: Expansion of supersonic wave
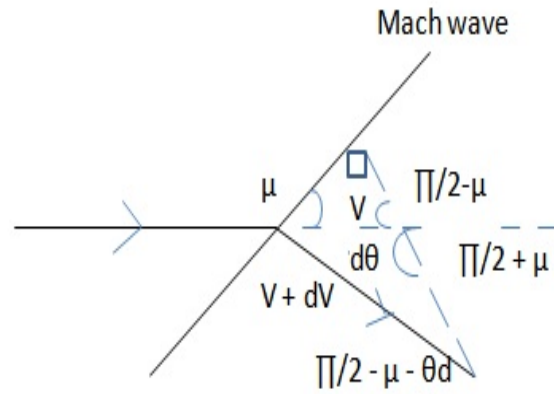
Figure 4: Velocity triangle across a typical Mach wave during supersonic expansion

$$\frac{V + dV}{V} = \frac{\sin\left(\frac{\pi}{2} + \mu\right)}{\sin\left(\frac{\pi}{2} - \mu - d\theta\right)}. \tag{65}$$

But,

$$\sin\left(\frac{\pi}{2} + \mu\right) = \sin\left(\frac{\pi}{2} - \mu\right) = \cos\mu, \tag{66}$$

and

$$\sin\left(\frac{\pi}{2} - (\mu + d\theta)\right) = \cos(\mu + \theta) = \cos\mu\cos d\theta - \sin\mu\sin d\theta. \tag{67}$$

Now we can write equation (65) as,

$$1 + \frac{dV}{V} = \frac{\cos\mu}{\cos\mu\cos d\theta - \sin\mu\sin d\theta}. \tag{68}$$

By the approximating $\sin d\theta \approx d\theta$ and $\cos d\theta \approx 1$, Equation (68) will simplified to,

$$1 + \frac{dV}{V} = \frac{\cos\mu}{\cos\mu - d\theta\sin\mu}, \tag{69}$$

$$1 + \frac{dV}{V} = \frac{1}{1 - d\theta\,\tan\mu}. \tag{70}$$

Since $d\theta\,\tan\mu < 1$, and using expansion for $x < 1$,

$$\frac{1}{1 - x} = x + x^2 + x^3 + ..., \tag{71}$$

by neglecting higher order term, equation (69) can be written as,

$$1 + \frac{dV}{V} = 1 + d\theta\,\tan\mu,$$

$$d\theta = \frac{dV/V}{\tan\mu} + \cdots . \tag{72}$$

But we know that

$$\mu = \sin^{-1}\frac{1}{M}\text{then, } \tan\mu = -\frac{1}{\sqrt{M^2-1}}. \tag{73}$$

Equation (72) becomes,

$$d\theta = \sqrt{M^2-1}\frac{dV}{V}. \tag{74}$$

6.1   Prandtl-Meyer Function

For a positive change in $d\theta$, results the change in volume dV which leads to expansion of waves. Equation (74) holds good for small angles of compression where we get negative change in volume dV, On integrating this formula for total expansion angle we get the downstream Mach number.

$$\int_{\theta_1}^{\theta_2} d\theta = \int_{M_1}^{M_2} \sqrt{M^2-1}\frac{dV}{V}. \tag{75}$$

Integrand in Mach number can be expressed as,

$$V = Ma. \tag{76}$$

Taking $\ln$ on both sides and differentiating we get,

$$\ln V = \ln M + \ln a, \tag{77}$$

$$\frac{dV}{V} = \frac{dM}{M} + \frac{da}{a}. \tag{78}$$

Using isentropic relation, second term in the right hand side of equation (78) can be expressed in terms of Mach number as,

$$\left(\frac{a_0}{a}\right)^2 = \frac{T_0}{T} = 1 + \frac{\gamma-1}{2}M^2,$$

$$a = a_0\left(1 + \frac{\gamma-1}{2}M^2\right)^{-1/2},$$

$$\frac{da}{a} = -\left(\frac{\gamma-1}{2}\right)M\left(1 + \frac{\gamma-1}{2}M^2\right)^{-1}dM. \tag{79}$$

Using equations (78) and (79) in equation (75) we get,

$$\int_{\theta_1}^{\theta_2} d\theta = \int_{M_1}^{M_2} \frac{\sqrt{M^2-1}}{\left(1 + \dfrac{\gamma-1}{2}M^2\right)}\frac{dM}{M},$$

$$\theta_2 = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \tan^{-1}\sqrt{M^2-1},$$

$$\theta = V(M), \tag{80}$$

where,

$$V(M) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \tan^{-1}\sqrt{M^2-1}. \tag{81}$$

Here V is called Prandtl-Meyer Function,

$$\theta = V(M_2) - V(M_1). \tag{82}$$

Therefore we can calculate deflection angle θ and upstream Mach number, by using these two we can calculate downstream Mach number [3].

7   Results and Discussions

We gained the knowledge about Ideal gas laws, gas dynamics or compressible flow and governing equations of gas dynamics. In addition to that we learnt about conservation forms of Euler equations and also to express them in differential form using governing equations. It extends our knowledge from the identification of the state of a system to define an equation of state by using three thermodynamic properties such as pressure, volume and internal energy and also to express other properties in terms of these three. Now we can predict the probability of inefficiency of an object by the knowledge of entropy which describes the disorder of that object and also we can express it as a function of specific internal energy, density and Boltzmann's constant. We learnt how the entropy and second law of thermodynamics are related. It made us to write the Euler equations for an isothermal flow and to express Euler equation in terms of flux function. Study of expansion wave help us to understand the expansion and compression of Mach wave. Knowledge of Prandtl-Meyer function help us to find downstream Mach wave using angle of deflection θ.

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Analysis of MHD Stagnation Point Flow in Porous Media using Haar wavelets

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Abstract: *We study the effect of heat source/sink and suction/injection on the MHD stagnation point flow and heat transfer over a stretching sheet in porous media. The governing partial differential equations are transformed into nonlinear coupled differential equations using similarity transformation. The numerical solutions are obtained using Haar wavelet collocation method. The quasilinearization technique is used to handle the nonlinearity. The basic idea of Haar wavelet collocation method is to convert the ordinary differential equation into a system of algebraic equations that involves a finite number of variables. The effect of suction parameter S, magnetic parameter M, permeability parameter Ω and Prandtl number Pr on the velocity and temperature profiles are studied and represented graphically.*

Keywords: *MHD stagnation point flow, Heat transfer, Stretching sheet, Porous media, Haar wavelets, Collocation points, Quasilinearization.*

AMS Subject Classification: *65T60, 76S05, 76W05.*

1 Introduction

The stagnation point flow in a plane was introduced by Hiemenz [1]. He considered a two-dimensional stagnation point flow problem on a stationary plate and used similarity transformation to reduce the Navier-stokes equation to nonlinear ordinary differential equations. Since then, the idea to consider different aspect of the stagnation point flow problems has attracted a lot of interest among researchers. Stagnation point flow has various applications like the cooling of nuclear reactors during emergency shutdown, the cooling of electronic devices, MHD generators and cooling of infinite metallic plates in a bath, hydrodynamic processes in engineering applications, metallurgical processes, such as drawing, annealing and tinning of copper wires [2, 3]. The hydromagnetic stagnation point flow and heat transfer have applications in boundary layer along material handling conveyers, aerodynamic extrusion of plastic sheet and blood flow problems.

Pop et al. [4] studied the radiation effects on the flow of an incompressible viscous fluid over a flat sheet near the stagnation point using Runge-Kutta method coupled with a shooting technique. Khan et al. [5] and Gul et al. [6] used homotopy perturbation method to investigate the effect of temperature dependent thermal conductivity and heat transfer from solids of rectangular, cylindrical and spherical shapes. Ishak et al. [7] investigated the steady two-dimensional MHD stagnation point flow towards the stretching sheet with variable surface temperature over a vertical flat plate numerically using the Keller-box scheme. They discussed the effects of the governing parameters on the flow field and heat transfer characteristics. Qi and Hong-Qing [8] investigated the steady two dimensional MHD stagnation point flow towards a stretching sheet with variable surface temperature. They obtained the analytic solution and convergence region

using homotopy analysis method. Sharma and Singh [9] investigated the effect of thermal conductivity and heat source/sink on flow of a viscous incompressible electrically conducting fluid in the presence of uniform transverse magnetic field and variable free stream near a stagnation point on a non-conducting stretching sheet, numerically using shooting method.

Hayat et al. [10] applied homotopy analysis method to obtain the analytical solutions of two-dimensional MHD stagnation point flow of an incompressible micropolar fluid over a non-linear stretching surface. Ali et al. [11] obtained the numerical solution of the steady MHD mixed convection stagnation point flow of an incompressible, viscous and electrically conducting fluid using an implicit finite difference scheme, for both assisting and opposing flows. They showed that dual solutions exist for a certain range of the buoyancy parameter in the case of opposing flow. Mahapatra et al. [12] studied the effect of uniform transverse magnetic field on the two dimensional stagnation point flow of an incompressible viscous electrically conducting fluid over a stretching surface, when the surface is stretched in its own plane with a velocity proportional to the distance from the stagnation point. Makinde and Charles [13] investigated the hydromagnetic stagnation flow of an incompressible viscous, electrically conducting fluid, towards a stretching sheet in the presence of axially increasing free stream velocity, numerically using the Newton-Raphson shooting method along with fourth-order Runge-Kutta method.

Yian et al. [14] studied the steady two dimensional MHD stagnation point flow of a viscous and electrically conducting fluid over a permeable stretching sheet and obtained the numerical solution using an implicit finite difference scheme. Ali et al. [15] carried out a numerical investigation of the steady laminar two dimensional nonlinear MHD stagnation point flow and heat transfer of an incompressible viscous fluid towards a stretching sheet, taking the effects of induced magnetic field, viscous dissipation and radiation into consideration. Al-sudais [16] carried out a numerical study using shooting method to investigate the effects of variable thermal conductivity and heat source/sink on steady two dimensional radiative MHD boundary layer flow of a viscous incompressible electrically conducting fluid in the presence of variable free stream near a stagnation point on a non-conducting stretching sheet. Makinde [17] obtained the hydromagnetic mixed convection stagnation point flow towards a vertical plate embedded in a highly porous medium with radiation and internal heat generation using shooting technique together with Runge-Kutta sixth order method.

Ibrahim et al. [18] employed shooting technique with Runge-Kutta fourth order method to analyze the effect of magnetic field, Brownian motion and thermophoresis on stagnation point flow and heat transfer due to a nanofluid towards a stretching sheet. Mohamed et al. [19] considered the mathematical modeling for stagnation point flow over a stretching surface with convective boundary conditions and obtained the numerical solution using shooting method. Rasekh et al. [20] studied the steady two dimensional MHD stagnation point flow towards a permeable stretching sheet with chemical reaction and obtained analytical solutions using optimal homotopy asymptotic method. Makinde et al. [21] analyzed the combined effects of buoyancy force, convective heating, Brownian motion, thermophoresis and magnetic field on stagnation point flow and heat transfer due to nanofluid flow towards a stretching sheet, numerically using Runge-Kutta fourth order method with shooting technique. Akbar et al. [22] studied the two dimensional tangent hyperbolic fluid flow towards a stretching sheet with a magnetic field numerically using fourth-fifth order Runge-Kutta Fehlberg method.

Akbar et al. [23] also determined the numerical solutions of the steady MHD two dimen-

sional stagnation point flow of an incompressible nanofluid towards a stretching cylinder, taking the effects of radiation and convective boundary conditions into account. Shateyi and Makinde [24] studied the steady stagnation point flow and heat transfer of an electrically conducted incompressible viscous fluid over a convectively heated and radially stretching disk surface in the presence of an external uniform magnetic field perpendicular to the plane of the disk. They obtained the numerical solutions using a spectral relaxation method with a Chebyshev spectral collocation method. Ramesh et al. [25] analyzed the effect of radiation on the flow near the two-dimensional stagnation point of an incompressible, viscous, electrically conducting dusty fluid towards stretching sheet numerically using Runge-Kutta Fehlberg fourth-fifth order method. Nadeem et al. [26] studied numerically the effects of MHD, elasticity and nanoparticles on the two dimensional boundary layer flow and the heat transfer of a Maxwell fluid past a stretching sheet.

Seini and Makinde [27] investigated the MHD boundary layer flow of viscous, incompressible and electrically conducting fluid near a stagnation point on a vertical surface with slip numerically using shooting method. Hajmohammadi et al. [28] discussed the flow and heat transfer of Cu-water and Ag-water nanofluids over a permeable flat plate with convective boundary conditions. Ibrahim and Makinde [29] studied the problem of double-diffusion on boundary layer flow and heat transfer induced due to a nanofluid over a stretching vertical sheet. They obtained the numerical solution using Keller-box finite difference scheme. Farooq et al. [30] investigated the effects of nonlinear thermal radiation, heat transfer, Brownian motion and thermophoresis on the MHD stagnation point flow of a viscoelastic nanofluid towards a stretching surface. Ibrahim and Haq [31] obtained the numerical solution of MHD stagnation point flow past a stretching sheet with convective heating and zero normal flux condition of nanoparticles at the wall. Khan et al. [32] studied the effects of chemical reaction on MHD stagnation point flow, heat and mass transfer of a Casson liquid past a stretched surface numerically using shooting method.

Bhatti et al. [33] used a combination of successive linearization method and Chebyshev spectral collocation method to study the effects of MHD and heat transfer on the stagnation point flow over a permeable stretching/shrinking sheet through porous media. Bhatti and Rashidi [34] also investigated the effect of entropy generation on the MHD stagnation point flow of a nanofluid over a permeable stretching surface using the same method. Agbaje et al. [35] investigated the MHD stagnation point flow and heat transfer from a stretching sheet in the presence of a heat source/sink and suction/injection in porous media, numerically using spectral perturbation method. They compared their results with those obtained using spectral quasilinearization method. Narayana et al. [36] examined the effects of viscous dissipation, joule heating, Brownian motion and thermophoresis on the MHD stagnation point flow of an incompressible viscoelastic nanofluid over a convectively heated stretching surface. Besthapu et al. [37] obtained a numerical solution of the MHD stagnation point flow of a Casson nanofluid past a convectively nonlinear stretching surface in porous medium using Keller box method. They studied the combined effects of thermal radiation and velocity slip on the flow. Ismail et al. [38] investigated the unsteady stagnation point flow and heat transfer past a shrinking sheet in the presence of suction and viscous dissipation.

In the year 1909, Alfred Haar, a Hungarian mathematician introduced Haar function which were later known as Haar wavelets. His contribution to wavelets is very evident. There is an entire wavelet family named after him. The Haar wavelet is a sequence of rescaled "square-

shaped" functions which together form a wavelet family or basis. They consist of piecewise constant functions and are therefore the simplest orthonormal wavelets with a compact support. An advantage of these wavelets is the possibility to integrate them analytically for arbitrary times. They are conceptually simple, fast, memory efficient and exactly reversible [39].

Lepik [41] solved ordinary and partial differential equations using Haar wavelet techniques. Yousefi [42] obtained the numerical solution of the Lane-Emden equations by converting these equations into integral equations using the integral operator and then applied the Legendre wavelets to convert the integral equations to algebraic equations. The algebraic equations are then solved using Gaussian integration method. Hsiao and Wu [43] used Haar wavelets to analyze time-varying functional differential equations. Bujurke et al. [44] solved elliptic boundary value problems arising in mathematical physics using Daubechies wavelet based multigrid technique. Lepik [45] presented the solution of higher order linear and nonlinear differential equations, both initial and boundary value problems, using Haar wavelets. Lepik [46] determined the solution of integral and differential equations using non-uniform Haar wavelets. Chang and Piau [47] used Haar wavelets to find the numerical solution of ordinary differential equations with constant and variable coefficients. Bujurke et al. [48] computed the eigenvalues and solutions of Sturm-Liouville eigenvalue problems using truncated Haar wavelet series.

Bujurke et al. [49] used single-term Haar wavelet series approach to obtain the solution of nonlinear stiff differential equations encountered in nonlinear dynamics. Bujurke et al. [50] also determined the solution of Duffing equation and Painleve's transcendents using single-term Haar wavelet series. Haq and Ali [51] obtained the numerical solution of multi-point fourth-order boundary value problems using uniform Haar wavelets. They compared their results with those determined using homotopy analysis method, differential transform method, Adomian decomposition method and homotopy perturbation method. Mohammadi and Hosseini [52] used Legendre wavelets to solve linear and nonlinear singular ordinary differential equations. Shi and Cao [53] computed the solution of eigenvalue problems of high-order ordinary differential equations using Haar wavelets. Khalid et al. [54] obtained the numerical solution of Airy differential equations using Haar wavelet method. Reddy et al. [55, 56] solved seventh and eighth order differential equations using Haar wavelets. Sumana et al. [57, 58, 59] determined the solution of non-homogeneous, non-planar and time-delayed Burgers' equations numerically using Haar wavelets.

In this chapter, we have studied the MHD stagnation point flow and heat transfer in porous media and obtained the numerical solutions using Haar wavelet collocation method.

2   Haar Wavelets

The Haar wavelet family [39] for $x \in [0, 1]$ is defined as follows,

$$
h_i(x) = \begin{cases} 1 & \text{for } x \in [\xi_1, \xi_2), \\ -1 & \text{for } x \in [\xi_2, \xi_3), \\ 0 & \text{elsewhere,} \end{cases} \tag{1}
$$

where

$$\xi_1 = \frac{k}{m}, \quad \xi_2 = \frac{k+0.5}{m}, \quad \xi_3 = \frac{k+1}{m}. \tag{2}$$

In the above definition $m = 2^d, d = 0, 1, ..., J$ indicates the level of the wavelet; $k = 0, 1, ..., m-1$ is the translation parameter. J is the maximum level of resolution. The index i in equation (1) is calculated by the formula $i = m + k + 1$. In the case of minimum values $m = 1, k = 0$ we have $i = 2$. The maximum value of i is $i = 2N = 2^{J+1}$. For $i = 1$, $h_1(x)$ is assumed to be the scaling function which is defined as follows.

$$h_1(x) = \begin{cases} 1 & \text{for } x \in [0,1), \\ 0 & \text{elsewhere.} \end{cases} \tag{3}$$

In order to solve differential equations of any order, we need the following integrals,

$$p_i(x) = \int_0^x h_i(x)dx = \begin{cases} x - \xi_1 & \text{for } x \in [\xi_1, \xi_2), \\ \xi_3 - x & \text{for } x \in [\xi_2, \xi_3), \\ 0 & \text{elsewhere,} \end{cases} \tag{4}$$

$$q_i(x) = \int_0^x p_i(x)dx = \begin{cases} \dfrac{(x-\xi_1)^2}{2} & \text{for } x \in [\xi_1, \xi_2), \\ \dfrac{1}{4m^2} - \dfrac{(\xi_3-x)^2}{2} & \text{for } x \in [\xi_2, \xi_3), \\ \dfrac{1}{4m^2} & \text{for } x \in [\xi_3, 1], \\ 0 & \text{elsewhere,} \end{cases} \tag{5}$$

$$r_i(x) = \int_0^x q_i(x)dx = \begin{cases} \dfrac{(x-\xi_1)^3}{6} & \text{for } x \in [\xi_1, \xi_2), \\ \dfrac{x-\xi_2}{4m^2} - \dfrac{(\xi_3-x)^3}{6} & \text{for } x \in [\xi_2, \xi_3), \\ \dfrac{x-\xi_3}{4m^2} & \text{for } x \in [\xi_3, 1], \\ 0 & \text{elsewhere.} \end{cases} \tag{6}$$

2.1   Function approximation

Any function $f(x)$ which is square integrable on $[0, 1)$ can be expressed as an infinite sum of Haar wavelets as

$$f(x) = \sum_{i=1}^{\infty} a_i h_i(x), \tag{7}$$

where

$$a_i = \int_0^1 f(x)h_i(x)dx. \tag{8}$$

If $f(x)$ is approximated as piecewise constant in each subinterval, then equation (7) will be terminated at finite terms, i.e.

$$f(x) = \sum_{i=1}^{2N} a_i h_i(x),\tag{9}$$

where the wavelet coefficients $a_i, \; i = 1, 2, \ldots, 2N$ are to be determined.

3   Formulation of the Problem

Consider the steady, two-dimensional MHD flow and heat transfer in porous media from an incompressible electrically conducting fluid in the neighbourhood of the stagnation point on a heated stretching surface with free stream velocity $U_\infty(x)$, uniform ambient temperature T_∞, stretching velocity $U_w(x)$ and surface temperature $T_w(x)$.

$$\begin{aligned} U_\infty(x) &= bx, \\ U_w(x) &= ax, \\ T_w(x) &= T_\infty + cx^n, \end{aligned}\tag{10}$$

where $a > 0$, $b \geq 0$, $c > 0$ and n are constants. A uniform magnetic field of strength B_0 is applied in the positive y-direction normal to the stretching sheet. The magnetic Reynolds number is assumed to be small, and thus the induced magnetic field is negligible. The flow model is shown in Figure 1.

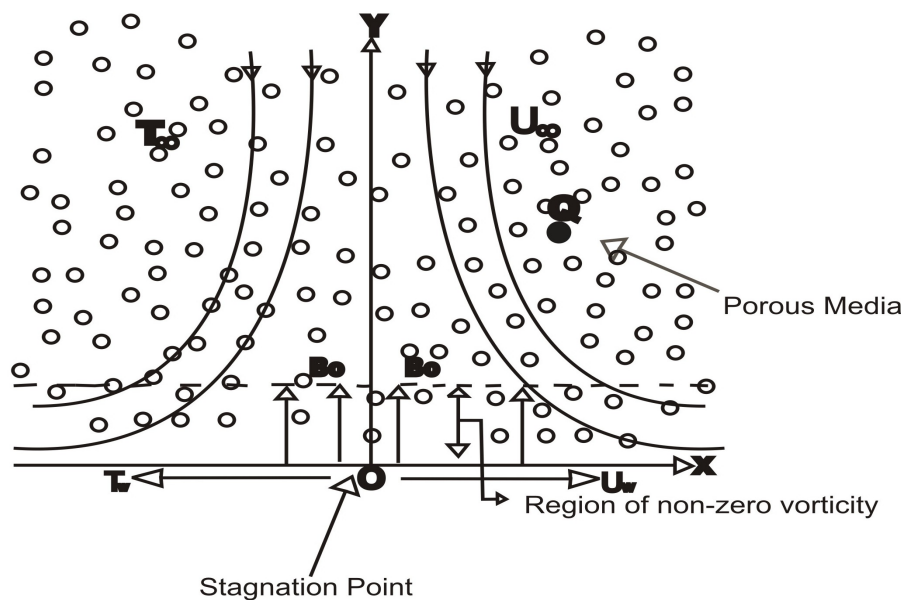

Figure 1: Flow Diagram.

The simplified boundary layer equations governing the flow and heat transfer are,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,\tag{11}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_\infty \frac{dU_\infty}{dx} + \nu \frac{\partial^2 u}{\partial y^2} + \frac{\sigma B_0^2}{\rho}(U_\infty - u) - \frac{\nu}{K}u,\tag{12}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{Q}{\rho C_p}(T - T_\infty), \tag{13}$$

subject to boundary conditions,

$$\left.\begin{aligned} u = U_w(x), \ v = V_w(x), \ T = T_w(x), \quad \text{at} \quad y = 0, \\ u \to U_w(x), \ T \to T_\infty \quad \text{as} \quad y \to \infty, \end{aligned}\right\} \tag{14}$$

where,
* u, v are velocity component along x, y respectively,
* T is the fluid temperature,
* ν is the kinematic viscosity,
* ρ is the fluid density,
* σ is the electrical conductivity,
* α is the thermal diffusivity,
* K is the constant permeability of the porous media,
* C_p is the specific heat at constant pressure,
* Q represents the temperature-dependent heat source/sink with $Q > 0$ for source and $Q < 0$ for sink, and
* $V_w(x)$ is the mass flux velocity with $V_w(x) < 0$ for suction and $V_w(x) > 0$ for injection.

Introduce a stream function ψ such that

$$u = \frac{\partial \psi}{\partial y}, \ v = -\frac{\partial \psi}{\partial x}, \tag{15}$$

which automatically satisfies the continuity equation (11).
The momentum (12) and energy (13) equations can be transformed into the corresponding nonlinear ordinary differential equations by the following similarity transformation.

$$\left.\begin{aligned} \eta &= y\sqrt{\frac{a}{\nu}}, \\ \psi &= x\sqrt{a\nu}f(\eta), \\ \theta(\eta) &= \frac{T - T_\infty}{T_w - T_\infty}, \end{aligned}\right\} \tag{16}$$

where η is the independent similarity variable.
The mass flux velocity $V_w(x)$ is assumed in the form,

$$V_w(x) = -S\sqrt{a\nu} \tag{17}$$

where S is the constant mass flux with $S > 0$ for suction and $S < 0$ for injection.
Form equations (16), we have

$$u = axf'(\eta), \tag{18}$$

$$v = -\sqrt{a\nu}f(\eta), \tag{19}$$

$$T = cx^n\theta(\eta) + T_\infty, \tag{20}$$

$$\frac{\partial u}{\partial x} = af'(\eta), \tag{21}$$

$$\frac{\partial u}{\partial y} = a\sqrt{\frac{a}{\nu}}xf''(\eta), \tag{22}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{a^2}{\nu} x f'''(\eta), \tag{23}$$

$$\frac{\partial T}{\partial x} = cnx^{n-1}\theta(\eta), \tag{24}$$

$$\frac{\partial T}{\partial y} = c\sqrt{\frac{a}{\nu}} x^n \theta'(\eta), \tag{25}$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{ac}{\nu} x^n \theta''(\eta). \tag{26}$$

Substituting equations (18)-(26) in equations (12)-(14), we get

$$f'''(\eta) + f(\eta)f''(\eta) - f'^2(\eta) - (M + \Omega)f'(\eta) + \epsilon(\epsilon + M) = 0, \tag{27}$$

$$\frac{1}{\text{Pr}}\theta''(\eta) + f(\eta)\theta'(\eta) + (\gamma - nf'(\eta))\,\theta(\eta) = 0, \tag{28}$$

subject to boundary conditions,

$$\left.\begin{array}{l} f(0) = S, \ f'(0) = 1, \ f'(\infty) = \epsilon, \\ \theta(0) = 1, \ \theta(\infty) = 0, \end{array}\right\} \tag{29}$$

where,

$\diamond$ $M = \dfrac{\sigma B_0^2}{\rho a}$ is the magnetic parameter,

$\diamond$ $\Omega = \dfrac{\nu}{Ka}$ is the permeability parameter,

$\diamond$ $\epsilon = \dfrac{b}{a}$ is the velocity ratio parameter,

$\diamond$ $\text{Pr} = \dfrac{\nu}{\alpha}$ is the Prandtl number, and

$\diamond$ $\gamma = \dfrac{Q}{a\rho C_p}$ is the dimensionless heat generation/absorption coefficient.

4   Method of Solution

In this section, the description of the Haar wavelet collocation method to solve equations (27)-(29) is outlined.

The order of the ODE (27) is 3 w.r.t. η. Therefore the **Haar Wavelet solution** is assumed to be in the form,

$$f'''(\eta) = \sum_{i=1}^{2N} a_i h_i(\eta). \tag{30}$$

Integrating equation (30) w.r.t. η in the limits $[0, \eta]$ gives

$$f''(\eta) = f''(0) + \sum_{i=1}^{2N} a_i p_i(\eta). \tag{31}$$

Integrating equation (31) w.r.t. η in the limits $[0, \eta]$ and using the conditions in (29) leads to

$$f'(\eta) = 1 + \eta f''(0) + \sum_{i=1}^{2N} a_i q_i(\eta). \tag{32}$$

Integrating equation (32) w.r.t. η in the limits $[0, \eta]$ and using the conditions in (29), we arrive at

$$f(\eta) = S + \eta + \frac{1}{2}\eta^2 f''(0) + \sum_{i=1}^{2N} a_i r_i(\eta). \tag{33}$$

Putting $\eta = L$ in equation (32) and using the conditions in (29), where L is sufficiently large integer, we get

$$f''(0) = \frac{\epsilon - 1}{L} - \frac{1}{L}\sum_{i=1}^{2N} a_i q_i(L). \tag{34}$$

Substituting equation (34) in equations (31)-(33), we obtain

$$f''(\eta) = \frac{\epsilon - 1}{L} + \sum_{i=1}^{2N} a_i \left[p_i(\eta) - \frac{1}{L} q_i(L) \right], \tag{35}$$

$$f'(\eta) = 1 + \left(\frac{\epsilon - 1}{L} \right)\eta + \sum_{i=1}^{2N} a_i \left[q_i(\eta) - \frac{\eta}{L} q_i(L) \right], \tag{36}$$

$$f(\eta) = S + \eta + \left(\frac{\epsilon - 1}{2L} \right)\eta^2 + \sum_{i=1}^{2N} a_i \left[r_i(\eta) - \frac{\eta^2}{2L} q_i(L) \right]. \tag{37}$$

Similarly, the order of the ODE (28) is 2 w.r.t. η and hence the **Haar Wavelet solution** is taken in the form,

$$\theta''(\eta) = \sum_{i=1}^{2N} b_i h_i(\eta). \tag{38}$$

Integrating equation (38) w.r.t. η in the limits $[0, \eta]$ gives

$$\theta'(\eta) = \theta'(0) + \sum_{i=1}^{2N} b_i p_i(\eta). \tag{39}$$

Integrating equation (39) w.r.t. η in the limits $[0, \eta]$ and using the conditions in (29) leads to

$$\theta(\eta) = 1 + \eta\theta'(0) + \sum_{i=1}^{2N} b_i q_i(\eta). \tag{40}$$

Putting $\eta = L$ in equation (40) and using the conditions in (29), we get

$$\theta'(0) = -\frac{1}{L} - \frac{1}{L}\sum_{i=1}^{2N} b_i q_i(L). \tag{41}$$

Substituting equation (41) in equations (39) and (40), we obtain

$$\theta'(\eta) = -\frac{1}{L} + \sum_{i=1}^{2N} b_i \left[p_i(\eta) - \frac{1}{L} q_i(L) \right], \tag{42}$$

$$\theta(\eta) = 1 - \frac{\eta}{L} + \sum_{i=1}^{2N} b_i \left[q_i(\eta) - \frac{\eta}{L} q_i(L) \right]. \tag{43}$$

Applying quasilinearization technique [60] to handle the nonlinear terms in equations (27) and (28), we have

$$
\begin{aligned}
f'''_{t+1}(\eta) + f_t(\eta)f''_{t+1}(\eta) &- \{2f'_t(\eta) + M + \Omega\}\, f'_{t+1}(\eta) + f''_t(\eta)f_{t+1}(\eta) \\
&= f_t(\eta)f''_t(\eta) - f'^2_t(\eta) - \epsilon(\epsilon + M),
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
\frac{1}{\mathrm{Pr}}\theta''_{t+1}(\eta) + f_t(\eta)\theta'_{t+1}(\eta) &+ \{\gamma - nf'_t(\eta)\}\,\theta_{t+1}(\eta) - n\theta_t(\eta)f'_{t+1}(\eta) \\
&+ \theta'_t(\eta)f_{t+1}(\eta) = f_t(\eta)\theta'_t(\eta) - nf'_t(\eta)\theta_t(\eta),
\end{aligned}
\tag{45}
$$

where t is the iteration parameter.

The wavelet collocation points are defined as,

$$
\eta_l = \frac{l - 0.5}{2N}, \quad l = 1, 2, \ldots, 2N.
\tag{46}
$$

Substituting equations (30), (35)-(38), (42)-(43) for $f'''_{t+1}(\eta)$, $f''_{t+1}(\eta)$, $f'_{t+1}(\eta)$, $f_{t+1}(\eta)$, $\theta''_{t+1}(\eta)$, $\theta'_{t+1}(\eta)$, $\theta_{t+1}(\eta)$ respectively in equations (44), (45) and taking $\eta \to \eta_l$, we get

$$
\sum_{i=1}^{2N} a_i A(i, l) = B(l),
\tag{47}
$$

$$
\sum_{i=1}^{2N} [a_i C(i, l) + b_i D(i, l)] = E(l),
\tag{48}
$$

where,

$$
\begin{aligned}
A(i, l) = h_i(\eta_l) + f_t(\eta_l)p_i(\eta_l) &- \{2f'_t(\eta_l) + M + \Omega\}\, q_i(\eta_l) + f''_t(\eta_l)r_i(\eta_l) \\
&- \frac{1}{L}\left[f_t(\eta_l) - \eta_l\{2f'_t(\eta_l) + M + \Omega\} + \frac{1}{2}\eta_l^2 f''_t(\eta_l)\right] q_i(L),
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
B(l) = f_t(\eta_l)f''_t(\eta_l) &- f'^2_t(\eta_l) - \epsilon(\epsilon + M) - \left(\frac{\epsilon - 1}{L}\right) f_t(\eta_l) \\
&+ \left\{1 + \left(\frac{\epsilon - 1}{L}\right)\eta_l\right\}\{2f'_t(\eta_l) + M + \Omega\} - \left\{S + \eta_l + \left(\frac{\epsilon - 1}{2L}\right)\eta_l^2\right\} f''_t(\eta_l),
\end{aligned}
\tag{50}
$$

$$
C(i, l) = \theta'_t(\eta_l)r_i(\eta_l) - n\theta_t(\eta_l)q_i(\eta_l) - \frac{1}{L}\left[\frac{1}{2}\eta_l^2\theta'_t(\eta_l) - n\eta_l\theta_t(\eta_l)\right] q_i(L),
\tag{51}
$$

$$
\begin{aligned}
D(i, l) = \frac{1}{\mathrm{Pr}}h_i(\eta_l) + f_t(\eta_l)p_i(\eta_l) &+ \{\gamma - nf'_t(\eta_l)\}\, q_i(\eta_l) \\
&- \frac{1}{L}\left[f_t(\eta_l) + \eta_l\{\gamma - nf'_t(\eta_l)\}\right] q_i(L),
\end{aligned}
\tag{52}
$$

$$
\begin{aligned}
E(l) = f_t(\eta_l)\theta'_t(\eta_l) - nf'_t(\eta_l)\theta(\eta_l) &+ \frac{1}{L}f_t(\eta_l) - \left(1 - \frac{\eta_l}{L}\right)\{\gamma - nf'_t(\eta_l)\} \\
&+ n\left\{(1 + \left(\frac{\epsilon - 1}{L}\right)\eta_l\right\}\theta_t(\eta_l) - \left\{S + \eta_l + \left(\frac{\epsilon - 1}{2L}\right)\eta_l^2\right\}\theta'_t(\eta_l).
\end{aligned}
\tag{53}
$$

The wavelet coefficients a_i and b_i, $i = 1, 2, \ldots, 2N$ can be calculated from equations (47) and (48) using the following initial approximation.

$$
\begin{aligned}
f_0(\eta_l) = S, \quad f'_0(\eta_l) = 1, \quad f''_0(\eta_l) = 0, \\
\theta_0(\eta_l) = 1, \quad \theta'_0(\eta_l) = 0.
\end{aligned}
\tag{54}
$$

These coefficients are then substituted in equations (30), (35)-(38), (42)-(43) to obtain the approximate solutions at the collocation points $\eta \to \eta_l$.
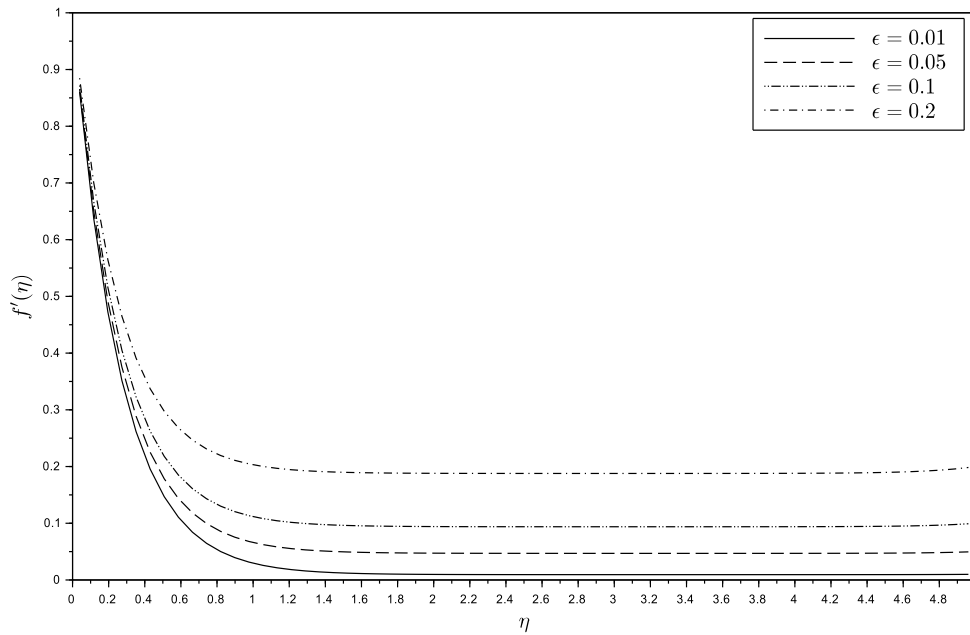
Figure 2: Effect of the velocity ratio parameter ϵ on the velocity profile when $S = -0.5$, $M = 15$ and $\Omega = 1$.
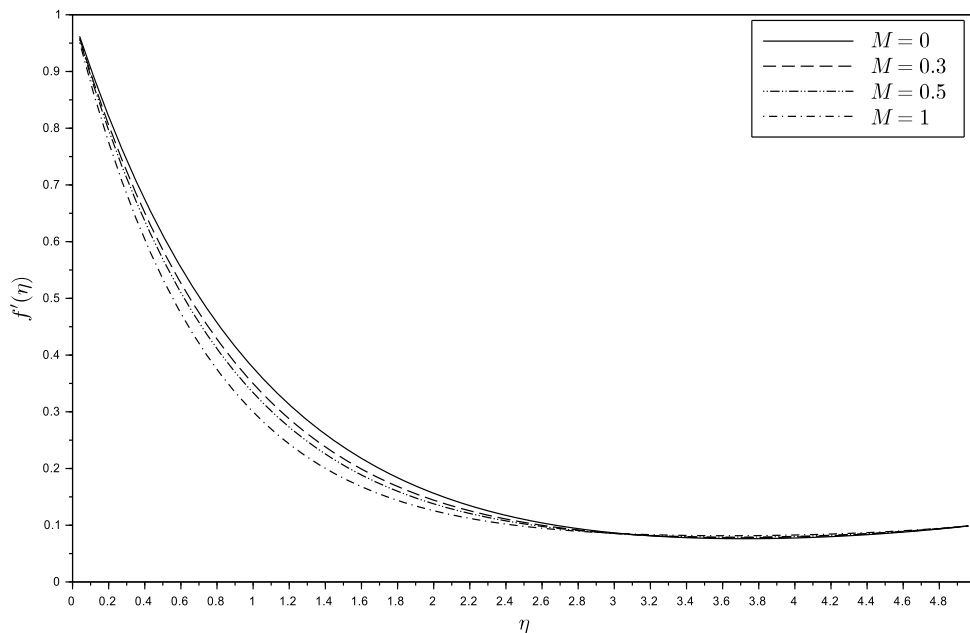
Figure 3: Effect of the magnetic parameter M on the velocity profile when $S = -0.5$, $\epsilon = 0.1$ and $\Omega = 0.5$.
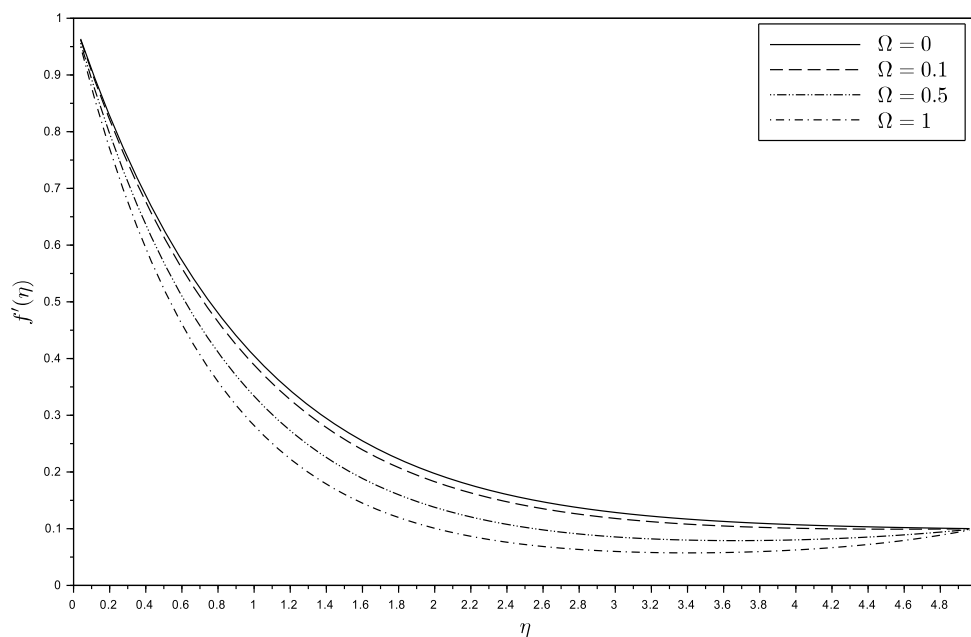
Figure 4: Effect of the permeability parameter Ω on the velocity profile when $S = -0.5$, $M = 0.5$ and $\epsilon = 0.1$.
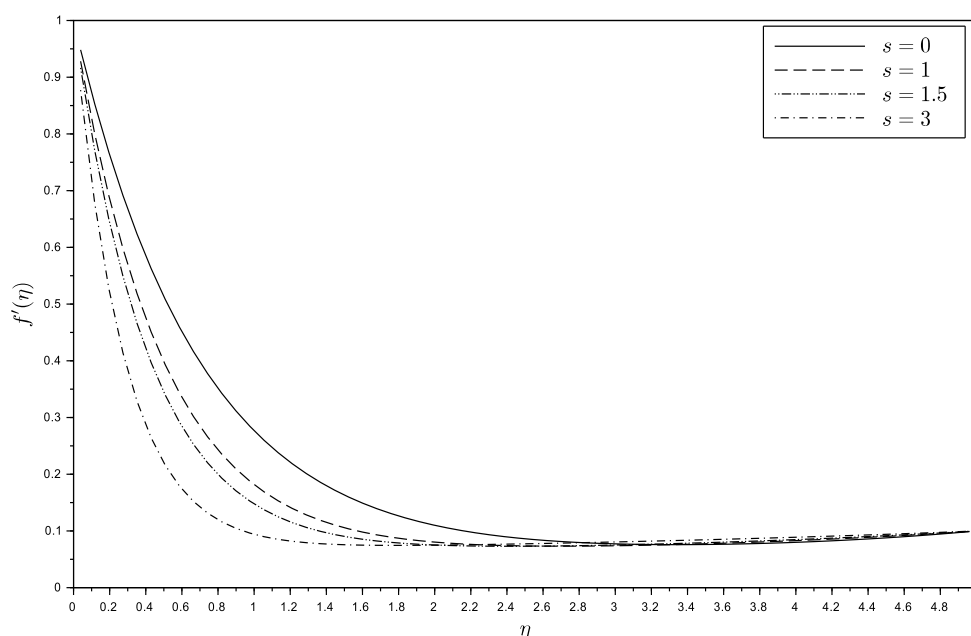
Figure 5: Effect of suction $(S > 0)$ on the velocity profile when $\Omega = 0.5$, $M = 0.5$ and $\epsilon = 0.1$.
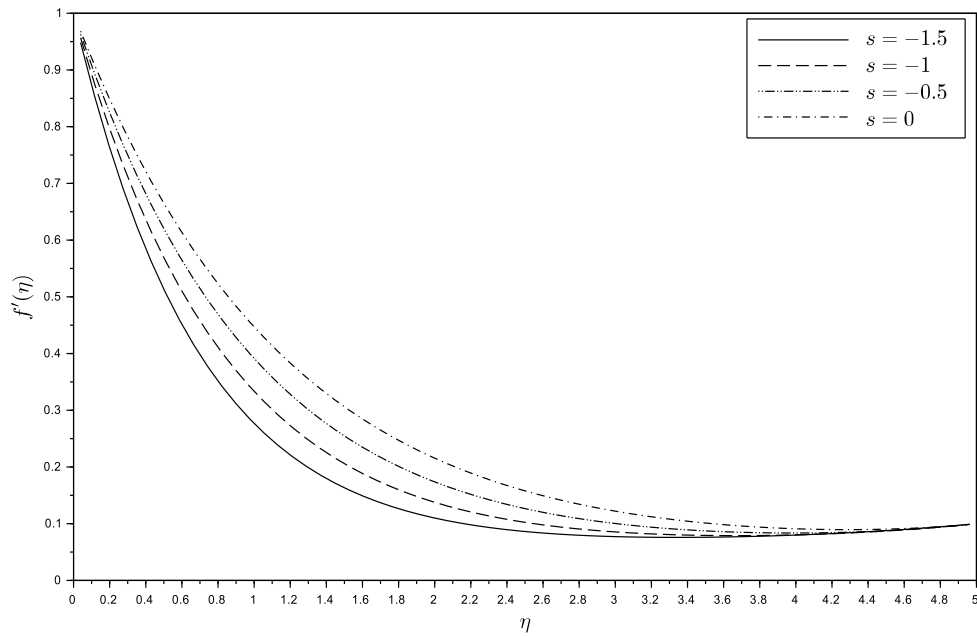
Figure 6: Effect of injection $S < 0$ on the velocity profile when $\Omega = 0.5$, $M = 0.5$ and $\epsilon = 0.1$.
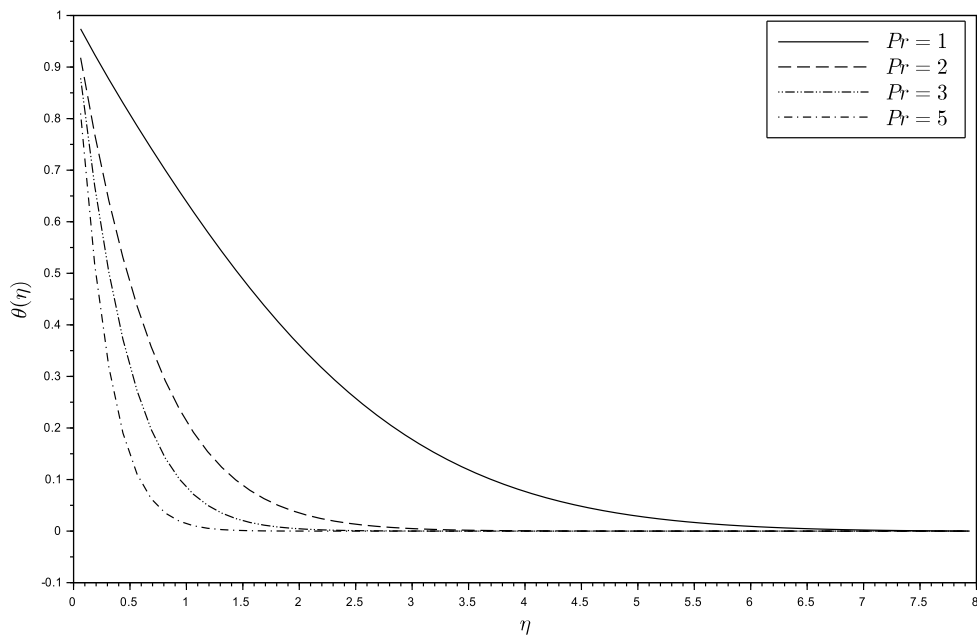
Figure 7: Effect of the Prandtl number Pr on the temperature profile when $S = -0.5$, $M = 0.5$, $\Omega = 0.5$, $n = 0.5$, $\gamma = 0.5$ and $\epsilon = 0.1$.
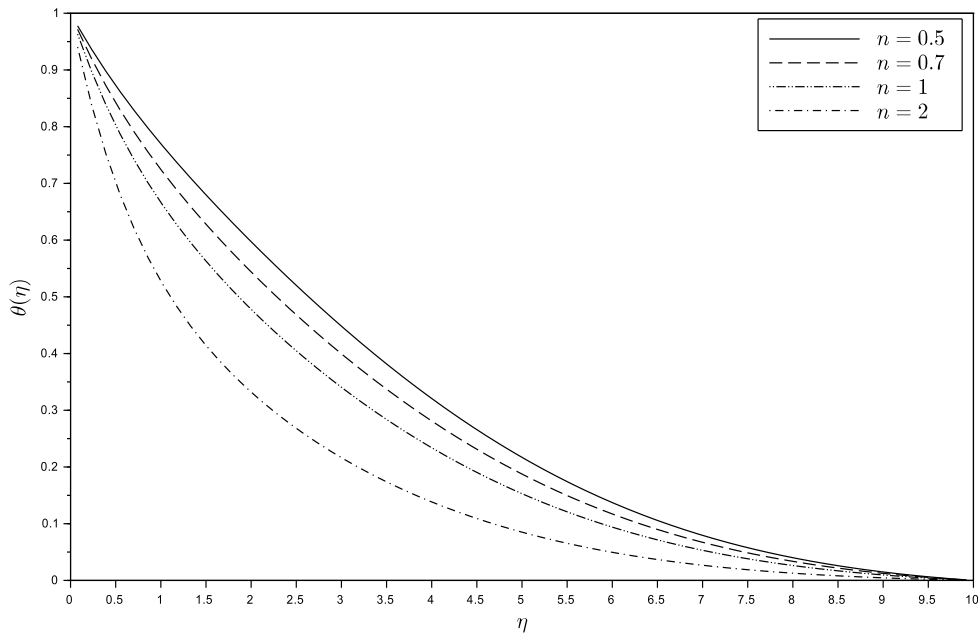
Figure 8: Effect of the constant n on the temperature profile when $S = -0.5$, $M = 0.5$, $\Omega = 0.5$, $\gamma = 0.1$, $\epsilon = 0.1$ and $\mathrm{Pr} = 0.7$.
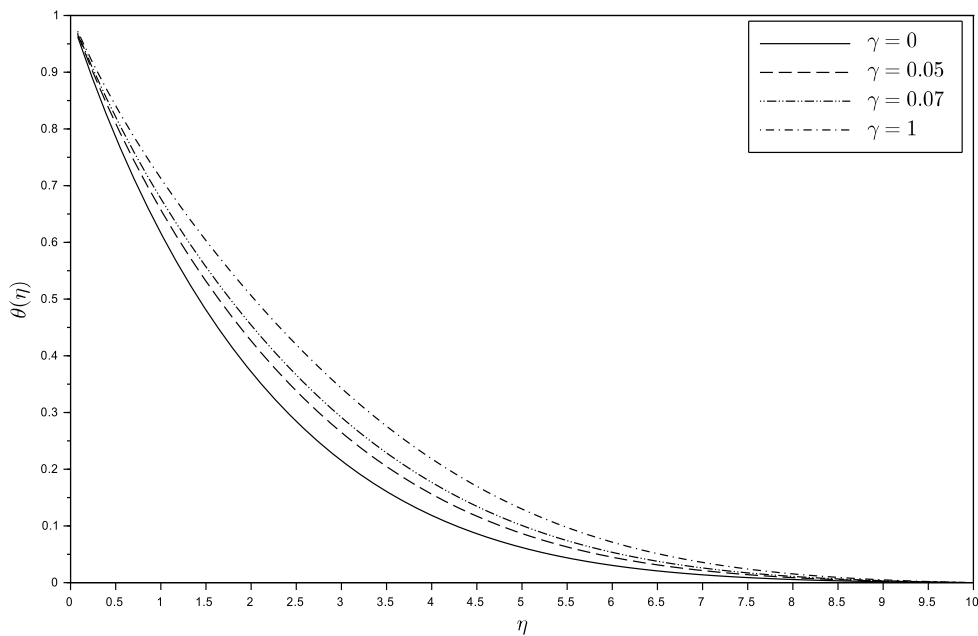
Figure 9: Effect of heat source ($\gamma > 0$) on the temperature profile when $S = -0.5$, $M = 0.5$, $\Omega = 0.5$, $n = 0.5$, $\epsilon = 0.1$ and $\mathrm{Pr} = 1$.
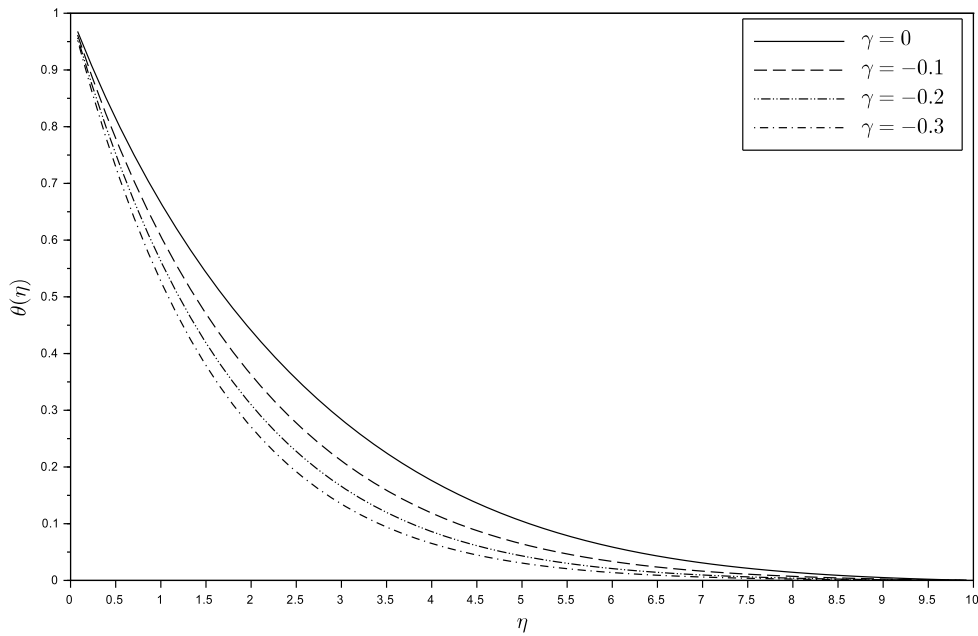
Figure 10: Effect of heat sink ($\gamma < 0$) on the temperature profile when $S = -0.5$, $M = 0.5$, $\Omega = 0.5$, $n = 0.5$, $\epsilon = 0.1$ and $\mathrm{Pr} = 0.75$.
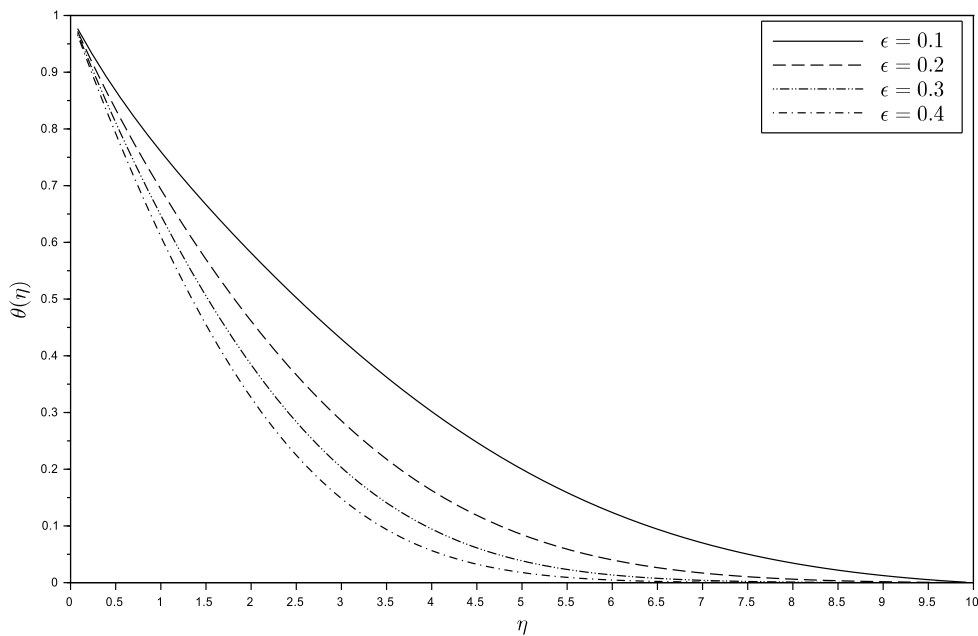
Figure 11: Effect of the velocity ratio parameter ϵ on the temperature profile when $S = -0.5$, $M = 0.5$, $\Omega = 0.5$, $\gamma = 0.1$, $n = 0.5$ and $\mathrm{Pr} = 0.75$.
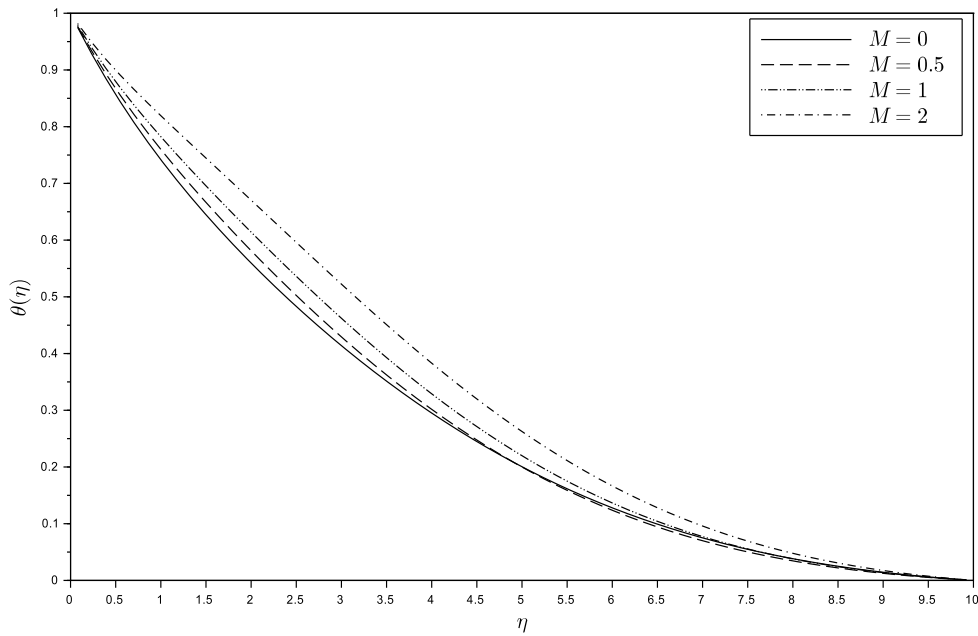
Figure 12: Effect of the magnetic parameter M on the temperature profile when $S = -0.5$, $\epsilon = 0.1$, $\Omega = 0.5$, $\gamma = 0.1$, $n = 0.5$, $\Pr = 0.75$.
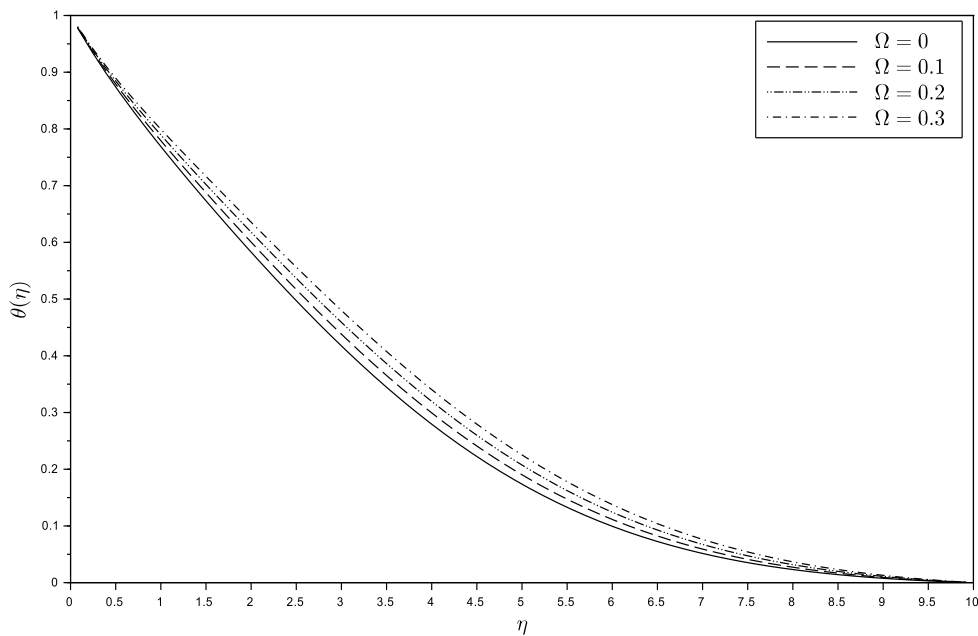
Figure 13: Effect of the permeability parameter Ω on the temperature profile when $S = -0.5$, $\epsilon = 0.1$, $M = 2$, $\gamma = 0.1$, $n = 0.5$ and $\Pr = 0.75$.

Figure 14: Effect of suction ($S > 0$) on the temperature profile when $\Omega = 0.3$, $\epsilon = 0.1$, $M = 2$, $\gamma = 0.1$, $n = 0.5$ and $\mathrm{Pr} = 0.75$.

Figure 15: Effect of injection ($S < 0$) on the temperature profile when $\Omega = 0.5$, $\epsilon = 0.1$, $M = 0.5$, $\gamma = 0.1$, $n = 0.5$ and $\mathrm{Pr} = 0.75$.
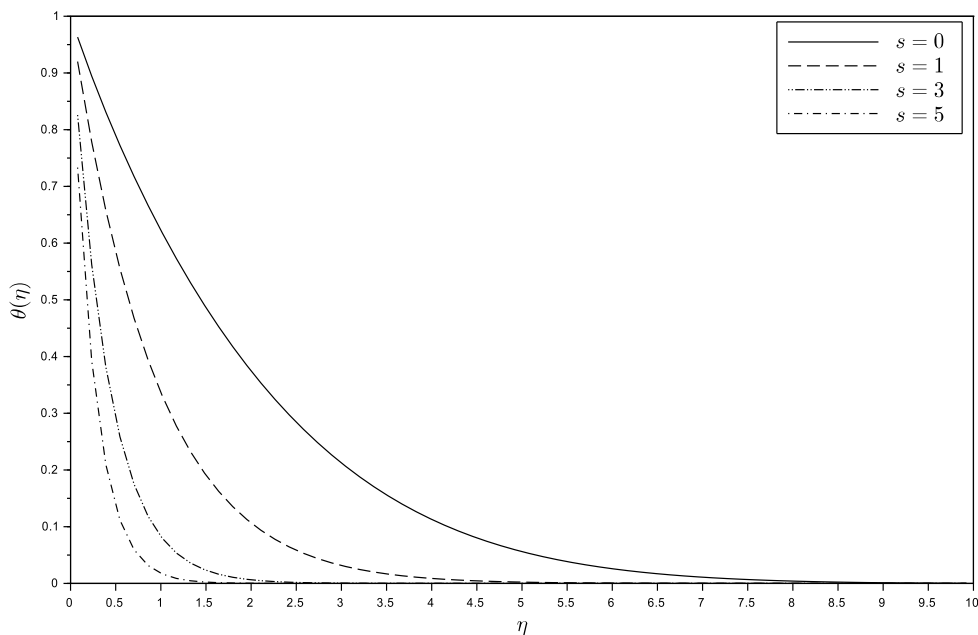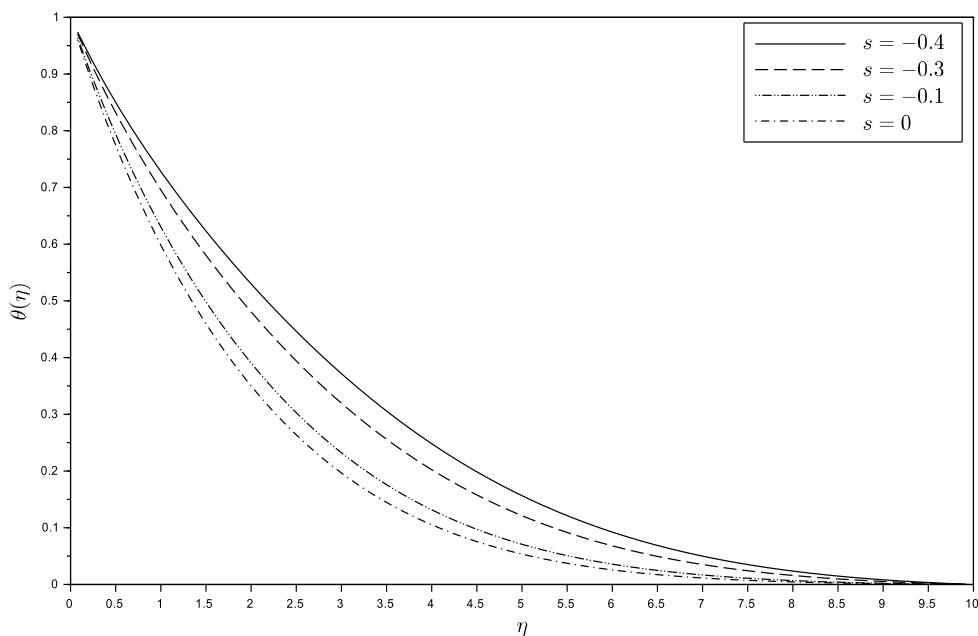
5   Results and Discussions

✳ Figure 2 shows the effect of the velocity ratio parameter ϵ on the velocity profile $f'(\eta)$. The velocity decreases with increase in the values of $\epsilon < 1$. This implies that there is a considerable decrease in the boundary layer thickness which leads to increase in ϵ. Increased value of ϵ means that the free stream velocity surpasses the stretching velocity.

✳ Figure 3 shows the effect of the Magnetic parameter M, on the velocity profile $f'(\eta)$. The velocity profile decreases with increase in M.

✳ Figure 4 illustrate the effect of the permeability parameter Ω on the velocity profile $f'(\eta)$. Ω increases as velocity profile decreases. This is due to the fact that the porous medium results into a drag force called Darcy force which decelerates the fluid in the boundary layer.

✳ Figure 5 shows the effect of suction S on the velocity profile $f'(\eta)$. The velocity decreases with increase in the value of S.

✳ Figure 6 shows the effect of injection S on the velocity profile $f'(\eta)$. The velocity decreases with increase in the value of S.

✳ Figure 7 presents the effect of the Prandtl number Pr on the temperature profile $\theta(\eta)$. Pr increases with decrease in $\theta(\eta)$. This is because of the fact that as the Pr increases, thermal boundary layer thickness reduces.

✳ Figure 8 presents the effect of the temperature index constant n on the temperature profile $\theta(\eta)$. $\theta(\eta)$ decreases with increase in n due to the decreasing manner of the thermal boundary layer thickness with increase in this parameter.

✳ Figure 9 shows the effect of the heat source generating parameter γ on the temperature profile $\theta(\eta)$. $\theta(\eta)$ decreases with increase in γ.

✳ Figure 10 presents the effect of the heat sink generating parameter γ on the temperature profile $\theta(\eta)$. $\theta(\eta)$ decreases with decrease in γ.

✳ Figure 11 shows the effect of velocity ratio parameter ϵ on the temperature profile $\theta(\eta)$. $\theta(\eta)$ decreases with increase in ϵ.

✳ Figure 12 shows the effect of Magnetic parameter M on the temperature profile $\theta(\eta)$. $\theta(\eta)$ decreases with increase in M.

✳ Figure 13 shows the effect of the Permeability parameter Ω on the temperature profile $\theta(\eta)$. $\theta(\eta)$ decreases with increase in Ω.

✳ Figure 14 shows the effect of suction S on the temperature profile $\theta(\eta)$. $\theta(\eta)$ decreases with increase in S.

✳ Figure 15 shows the effect of injection S on the temperature profile $\theta(\eta)$. $\theta(\eta)$ decreases with increase in S.

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Introduction to Z Transforms and its Applications

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Abstract: *Integral transform provides powerful operational method for solving initial value problem and initial boundary value problem for linear differential equations, difference equations, integral equations and many more arising problems in applied mathematics, mathematical physics and engineering science. In this paper we study about a type of integral transforms called Z transform, examples and its basic operational properties. Application of Z transform to difference equations and to infinite series are discussed with examples.*

Keywords: *Integral transforms, Z transforms, Finite difference equations, Summation of infinite series.*

AMS Subject Classification: 44A15.

1   Introduction

The integral transform of function $f(x)$ defined for $x \in [a, b]$ is defined by [1]

$$I\{f(x)\} = F(t) = \int_a^b K(x,t)f(x)\,dx$$

where $K(x,t)$ is called the Kernel of the transform, $F(t)$ is called the transformation function and t the transform variable.

The basic aim of the integral transform method is to transform a given problem into one that is easier to solve. When an ordinary differential equation with constant coefficients is considered, the given problem is transformed into an ordinary differential equation with algebraic terms. But when a partial differential equation with n independent variables is considered, it is transformed into a partial differential equation with $n - 1$ variables. Inversion of this solution produces the solution of the original problem [2, 3].

Two of the most famous and widely used integral transforms are the Fourier transforms and the Laplace transforms. In addition there are other transforms like the Hilbert transform and the Sturm-Liouville transform, both of which are limited in their fields of applications compared to the Fourier and Laplace transforms. A few discrete transforms are the Fourier transform (which is the discrete analog of the Fourier transform) and the Z transform (which is the discrete analog of the Laplace transform) which are being used in many engineering applications where it is either impossible or inconvenient to use more conventional transforms.

The Z transform is a powerful mathematical tool used in solving difference equations. The concept of Z transforms is similar to the concept of generating functions, as it is an operation

that converts a discrete signal into a complex frequency domain representation. The Z transform is also similar to the Laplace transform and thus has many common properties in between them. In the Laplace transforms we consider the time function as discontinuous, but in Z transform the time function is considered as a continuous function $x(t)$. With the usage of digital computers, more enhanced research has been done on the development of applications of Z transforms [4, 5].

Definition of Z Transform and inverse Z Transform

The Z transform of a sequence $\{f(n)\}$ is defined by [6, 7]

$$Z\{f(n)\} = F(z) = \sum_{n=0}^{\infty} f(n)z^{-n}$$

The inverse of Z transform is defined as

$$Z^{-1}\{f(z)\} = f(n) = \frac{1}{2\pi i} \oint_C F(z)z^{n-1}dz$$

2   Properties of Z Transforms

Theorem 1 (Translation or Shifting Property). *If $Z\{f(n)\} = F(z)$ and $m \geq 0$, then*

(i) $Z\{f(n-m)\} = z^{-m}\left[F(z) + \sum_{r=-m}^{-1} f(r)z^{-r}\right]$ *(Shifting to the right)*

(ii) $Z\{f(n+m)\} = z^m\left[F(z) - \sum_{r=0}^{m-1} f(r)z^{-r}\right]$ *(Shifting to the left)*

Proof. (i) Substitute $n - m = r$ in the definition of Z transforms, then we have

$$Z\{f(n-m)\} = z^{-m} \sum_{r=-m}^{\infty} f(r)z^{-r}$$

$$= z^{-m}\left[\sum_{r=-m}^{-1} f(r)z^{-r} + \sum_{r=0}^{\infty} f(r)z^{-r}\right]$$

$$= z^{-m}\left[\sum_{r=-m}^{-1} f(r)z^{-r} + F(z)\right]$$

(ii) Substitute $n + m = r$ in the definition of Z transforms, then we have

$$Z\{f(n+m)\} = z^m \sum_{r=m}^{\infty} f(r)z^{-r}$$

$$= z^m\left[\sum_{r=0}^{\infty} f(r)z^{-r} - \sum_{r=0}^{m-1} f(r)z^{-r}\right]$$

$$= z^m \left[F(z) - \sum_{r=0}^{m-1} f(r) z^{-r} \right]$$

$\square$

Theorem 2 (Multiplication Property). *If $Z\{f(n)\} = F(z)$, then*

    (i) $Z\{a^{-n} f(n)\} = F(az)$ *(ii)* $Z\{a^n f(n)\} = F\left(\dfrac{z}{a}\right)$

    (iii) $Z\{nf(n)\} = -z\dfrac{d}{dz} F(z)$

Proof. By the definition of Z transforms, we have

(i) $\begin{aligned}[t] Z\{a^{-n} f(n)\} &= \sum_{n=0}^{\infty} f(n)(az)^{-n} \\ &= F(az) \end{aligned}$

(ii) $\begin{aligned}[t] Z\{a^n f(n)\} &= \sum_{n=0}^{\infty} f(n)\left(\frac{z}{a}\right)^{-n} \\ &= F\left(\frac{z}{a}\right) \end{aligned}$

(iii) $\begin{aligned}[t] Z\{nf(n)\} &= z\sum_{n=0}^{\infty} nf(n) z^{-(n+1)} \\ &= -z\frac{d}{dz}\left\{ \sum_{n=0}^{\infty} f(n) z^{-n} \right\} \\ &= -z\frac{d}{dz} F(z) \end{aligned}$

$\square$

Theorem 3 (Division Property). *If $Z\{f(n)\} = F(z)$ then $Z\left\{\dfrac{f(n)}{n+m}\right\} = -z^m \displaystyle\int_0^z \dfrac{F(t)dt}{t^{m+1}}$*

Proof. By the definition of Z transforms we have $Z\left\{\dfrac{f(n)}{n+m}\right\} = \displaystyle\sum_{n=0}^{\infty} \dfrac{f(n)}{n+m} z^{-n}$

The right hand side of the above equation can be written as

$$Z\left\{\frac{f(n)}{n+m}\right\} = -z^m \int_0^z t^{-(m+1)} \left[\sum_{n=0}^{\infty} f(n) t^{-n} \right] dt$$

$$= -z^m \int_0^z \frac{F(t)dt}{t^{m+1}}$$

$\square$

Theorem 4 (Convolution Theorem). *If $Z\{f(n)\} = F(z)$ and $Z\{g(n)\} = G(z)$, then*

$$Z\{f(n) * g(n)\} = Z\{f(n)\}\, Z\{g(n)\}$$

where the convolution is defined by $f(n) * g(n) = \sum_{m=0}^{\infty} f(n-m)g(m)$. [5]

Proof. By the definition of convolution of two integral function we have,

$$Z\{f(n) * g(n)\} = \sum_{m=0}^{\infty} g(m) \sum_{n=0}^{\infty} f(n-m)z^{-n}$$

Substitute $n - m = r$ in the above expression, we obtain

$$Z\{f(n) * g(n)\} = \sum_{m=0}^{\infty} g(m)z^{-m} \sum_{r=-m}^{\infty} f(r)z^{-r} \quad \text{where } f(r) = 0 \text{ for } r < 0$$

$$= \sum_{m=0}^{\infty} g(m)z^{-m} \sum_{r=0}^{\infty} f(r)z^{-r}$$

$$= Z\{f(n)\}\, Z\{g(n)\}$$

□

3   Examples on Z Transforms

Example 1: Find the Z transform of the following functions: [8, 9]

(i) $f(n) = (n-1)^2$ (ii) $f(n) = \dfrac{(n+1)a^n}{n!}$

By the definition of Z transformation we have

(i) $Z\{f(n)\} = Z(n^2 - 2n + 1)$

$\qquad\quad = Z(n^2) - 2Z(n) + Z(1)$

$\qquad\quad = \dfrac{z^3 - 3z^2 + 4z}{(z-1)^3}$

(ii) $Z\{na^n\} = -z\dfrac{d}{dz}[Z\{a^n\}]$

$\qquad\quad = -z\dfrac{d}{dz}\left[\dfrac{z}{z-a}\right]$

$\qquad\quad = \dfrac{za}{(z-a)^2}$

Then $Z\{n^2 a^n\} = -z\dfrac{d}{dz}[Z\{na^n\}]$

$\qquad\qquad\quad = -z\dfrac{d}{dz}\left[\dfrac{za}{(z-a)^2}\right]$

$$= \frac{az(z+a)}{(z-a)^3}$$

Example 2: Find the inverse of Z transforms of the following functions:

(i) $F(z) = \dfrac{z^3}{(z^2-1)(z-2)}$ (ii) $F(z) = \dfrac{7z - 11z^2}{(z-1)(z-2)(z+3)}$

(i) Rewriting the given problem as $\dfrac{F(z)}{z} = \dfrac{z^2}{(z-1)(z+1)(z-2)}$

On solving the R.H.S. we obtain

$$\frac{z^2}{(z-1)(z+1)(z-2)} = \frac{-1}{2(z-1)} + \frac{1}{6(z+1)} + \frac{4}{3}\frac{1}{(z-2)}$$

Taking inverse Z transforms on both sides we have,

$$f(n) = \frac{1}{6}\left[(-1)^n - 3(1)^n + 2^3(2)^n\right]$$

(ii) Rewriting the given problem as $\dfrac{F(z)}{z} = \dfrac{7 - 11z}{(z-1)(z-2)(z+3)}$

On solving the R.H.S. we obtain

$$\frac{F(z)}{z} = \frac{1}{(z-1)} + \frac{-3}{(z-2)} + \frac{2}{(z+3)}$$

Taking inverse Z transforms on both sides we obtain

$$f(n) = 1 - 3(2)^n + 2(-3)^n$$

Example 3: Prove the following:[10]

(i) $Z\{\cosh(na)\} = \dfrac{z(z - \cosh(a))}{z^2 - 2z\cosh(a+1)}$ (ii) $Z\{\cos(bn)\} = \dfrac{z^2 - z\cos(b)}{z^2 - 2z\cos(b) + 1}$

(iii) $Z\left\{\dfrac{f(n)}{n+m}\right\} = z^m \displaystyle\int_z^\infty \dfrac{F(t)dt}{t^{m+1}}$

By the definition of Z transform we have

(i) $Z(\cosh(na)) = \displaystyle\sum_{n=0}^\infty \cosh(na)z^{-n}$

$$= \frac{1}{2}\sum_{n=0}^\infty e^{na}z^{-n} + \frac{1}{2}\sum_{n=0}^\infty e^{-na}z^{-n}$$

$$= \frac{1}{2}\left[\frac{1}{1 - e^a z^{-1}}\right] + \frac{1}{2}\left[\frac{1}{1 - e^{-a}z^{-1}}\right]$$

$$= \frac{z(z - \cosh(a))}{z^2 - 2z\cosh(a+1)}$$

(ii) We know that

$$Z\{a^n\} = \frac{z}{z - a} \tag{1}$$

Taking $a = e^{ib}$ in equation (1), we obtain

$$Z\{e^{ib}\} = \frac{z(z - e^{-ib})}{(z - e^{-ib})(z - e^{-ib})}$$

$$= \frac{z^2 - ze^{-ib}}{z^2 - z(e^{ib} + e^{-ib}) + 1}$$

$$Z\{\cos(bn)\} + iZ\{\sin(bn)\} = \frac{z^2 - z\cos(b)}{z^2 - 2z\cos(b+1)} + i\frac{z\sin(b)}{z^2 - 2z\cos(b+1)} \tag{2}$$

Equating the real and imaginary parts of equation (2), we have

$$Z\{\cos(bn)\} = \frac{z^2 - z\cos(b)}{z^2 - 2z\cos(b+1)}$$

$$Z[\sin(bn)] = \frac{z\sin(b)}{z^2 - 2z\cos(b+1)} \tag{3}$$

(iii) $Z\left\{\dfrac{f(n)}{n+m}\right\} = \displaystyle\sum_{n=0}^{\infty} \dfrac{f(n)}{n+m} z^{-n}$

Rewriting the R.H.S. of the above equation, we have

$$Z\left\{\frac{f(n)}{n+m}\right\} = z^m \sum_{n=0}^{\infty} f(n) \left[\int_z^{\infty} t^{-(n+m+1)} dt \right]$$

$$= z^m \int_z^{\infty} t^{-(m+1)} \left[\sum_{n=0}^{\infty} f(n) t^{-n} \right] dt$$

$$= z^m \int_z^{\infty} \frac{F(t) dt}{t^{m+1}}$$

4   Applications of of Z Transforms

Analyzing the behavior of functions whose values are known on a finite or infinite set of discrete points in a given domain cannot be achieved by using the Fourier or the Laplace Transform techniques. In signal processing system, to determine the output relation when the discrete input function is known can be solved using techniques in the Z transform.

4.1   Applications of Z Transforms to Finite Difference Equations

Difference equations arise in a variety of applications. They are closely related to differential equations. A linear finite difference equation with constant coefficients of the form $af(n+2)+bf(n+1)+cf(n) = \phi(n)$ with given values of $f(0)$ and $f(1)$, can be solved using Z transforms. We follow the following steps: [10]

1. Let $F(z) = Z\{f(n)\}$.
2. Apply Z transform on both sides of the given difference equation. To express $Z\{f(n+1)\}, Z\{f(n+2)\}, Z\{f(n+3)\}$, use the following results

   - $Z\{f(n+1)\} = zF(z) - zf(0)$
   - $Z\{f(n+2)\} = z^2 F(z) - z^2 f(0) - zf(1)$
   - $Z\{f(n+3)\} = z^3 F(z) - z^3 f(0) - z^2 f(1) - zf(2)$ and so on ......

3. Substitute the values of $f(0)$ and $f(1)$ and obtain an expression for $F(z)$.
4. Take the inverse Z transform of $F(z)$ and determine $f(n)$, which is the required solution.

Example 4: Solve the following difference equations using Z transforms:

(i) $f(n+2) + 3f(n+1) + 2f(n) = 0;\ f(0) = 1,\ f(1) = 2$

(ii) $f(n+1) + 3f(n) = n;\ f(0) = 1$

(iii) $f(n+2) - 5f(n+1) + 6f(n) = 2^n;\ f(0) = 1,\ f(1) = 0$

(i) Applying Z transforms on the both sides of given equation, we have

$$z^2[F(z) - f(0) - f(1)z^{-1}] + 3z[F(z) - f(0)] + 2F(z) = 0$$

$$\frac{F(z)}{z} = \frac{-(z+5)}{(z+2)\,(z+1)}$$

$$\frac{F(z)}{z} = \frac{-3}{(z+2)} + \frac{4}{(z+1)}$$

$$Z\{f(n)\} = -3\left[\frac{z}{z+2}\right] + 4\left[\frac{z}{z+1}\right]$$

Taking inverse Z transforms on both sides, we get

$$f(n) = -3(-2)^n + 4(-1)^n$$

(ii) Applying the Z transforms on the both sides of given equation, we have

$$z[F(z) - f(0)] + 3F(z) = \frac{z}{(z-1)^2}$$

$$\frac{F(z)}{z} = \frac{1}{(z+3)} + \frac{1}{(z+3)\,(z-1)^2}$$

$$Z\{f(n)\} = \frac{z}{(z+3)} + \frac{1}{16}\frac{z}{(z+3)} - \frac{z}{16(z-1)} + \frac{z}{4(z-1)^2}$$

Taking inverse Z transforms on both sides, we get

$$f(n) = \frac{1}{16}\{17(-3)^n - 1 + 4n\}$$

(iii) Applying the Z transforms on the both sides of given equation, we obtain

$$z^2[F(z) - f(0) - f(1)z^{-1}] - 5z[F(z) - f(0)] + 6F(z) = \frac{z}{z-2}$$

$$\frac{F(z)}{z} = \frac{1}{(z-2)^2\,(z-3)} + \frac{z-5}{(z-3)\,(z-2)}$$

$$\frac{F(z)}{z} = \left[\frac{3}{z-2} - \frac{2}{z-3}\right] + \left[\frac{1}{z-3} - \frac{1}{z-2} - \frac{1}{(z-2)^2}\right]$$

$$Z\{f(n)\} = 2\left[\frac{z}{z-2}\right] - \left[\frac{z}{z-3}\right] - \left[\frac{z}{(z-2)^2}\right]$$

Taking inverse Z transforms on both sides, we get

$$f(n) = 2^{n+1} - 3^n - n2^{n-1}$$

4.2   Applications of Z Transforms Summation of Infinite Series

Theorem 1. *If $Z\{f(n)\} = F(z)$, then*

(a) $\displaystyle\sum_{k=1}^{n} f(k) = Z^{-1}\left\{\frac{z\,F(z)}{z-1}\right\}$
(b) $\displaystyle\sum_{k=1}^{n} f(k) = \lim_{z\to 1} F(z) = F(1)$

Proof. **(b)** Consider $g(n) = \displaystyle\sum_{k=0}^{n} f(k)$

Then $g(n) = f(n) + g(n-1)$
Applying the Z transform to the above equation, we obtain

$$G(z) = F(z) + z^{-1}G(z)$$

$$G(z) = \frac{z\,F(z)}{(z-1)}$$

$$Z\left\{\sum_{k=0}^{n} f(k)\right\} = \frac{z\,F(z)}{(z-1)}$$

(b) Applying the limit as $z \to 1$ together with the final value theorem , we get

$$\lim_{n\to\infty}\sum_{k=0}^{n} f(k) = \lim_{z\to 1}(z-1)\frac{zF(z)}{z-1}$$

$$= F(1) \tag{4}$$

$\square$

Example 5: Find the sum of the following series using the Z transform :

(i) $\displaystyle\sum_{n=0}^{\infty} a^n e^{inx}$
(ii) $\displaystyle\sum_{n=0}^{\infty} a^n \sin(nx)$

(i) By the definition of Z transforms, we have

$$Z\{e^{inx}\} = \sum_{n=0}^{\infty} e^{inx} z^{-n}$$

$$= 1 + \frac{e^{ix}}{z} + \frac{e^{2ix}}{z^2} + \ldots\ldots$$

$$= \frac{z}{z - e^{ix}}$$

Then $Z\{a^n e^{inx}\} = \dfrac{\left(\dfrac{z}{a}\right)}{\left(\dfrac{z}{a}\right) - e^{ix}}$

$$= \frac{z}{z - ae^{ix}}$$

$$\sum_{n=0}^{\infty} a^n e^{inx} = \lim_{z \to 1} F\left(\frac{z}{a}\right) \quad \text{[Using equation (4)]}$$

$$= (1 - ae^{ix})^{-1}$$

(ii) By using equation (3) and the above result, we see that

$$Z\{a^n \sin(nx)\} = F\left(\frac{z}{a}\right)$$

$$= \frac{az \sin(x)}{a^2 - 2az \cos(x) + z^2}$$

$$\sum_{n=0}^{\infty} a^n \sin(nx) = \lim_{z \to 1} F\left(\frac{z}{a}\right)$$

$$= \frac{a \sin(x)}{a^2 - 2a \cos(x) + 1}$$

Example 6: Prove the following results:

(i) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \log(1+x)$ 　　　　(ii) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

(i) Consider $Z\{f(n)\} = \displaystyle\sum_{n=0}^{\infty} x^{n+1} z^{-n}$

$$= x\left[1 + \frac{x}{z} + \frac{x^2}{z^2} + \frac{x^3}{z^3} + \ldots\ldots\right]$$

$$= \frac{xz}{z - x}$$

By the Division property of Z transforms, we have

$$Z\left\{\frac{x^{n+1}}{n+1}\right\} = z \int_z^{\infty} \frac{zx}{(z-x)} \frac{dz}{z^2}$$

$$= zx \left[\frac{1}{x} \log\left(\frac{z-x}{z}\right)\right]_z^{\infty}$$

$$= -z \left[\log \left(\frac{z - x}{z} \right) \right]$$

Replacing x by $-x$ in the above result, we get

$$Z \left\{ (-1)^n \frac{x^{n+1}}{n+1} \right\} = z \log \left(\frac{z + x}{z} \right)$$

Using equation (4), we have

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \log(1 + x)$$

(ii) We have $Z\{x^n f(n)\} = F\left(\frac{z}{x} \right)$

$$Z \left\{ \frac{x^n}{n!} \right\} = \sum_{n=0}^{\infty} \frac{x^n}{n!} z^{-n} \qquad \left(\text{Since } F(z) = e^{\frac{1}{z}} \right)$$

$$= 1 + \frac{x}{z} + \frac{x^2}{2!z^2} + \frac{x^3}{3!z^3} + \cdots \cdots$$

$$= e^{\frac{x}{z}}$$

Using equation (4), we have

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

5　Conclusion

We have studied about a type of integral transform called Z transforms, its basic properties and various theorems on it. We have also seen how to apply Z transforms on finite difference equations and to find the summation of infinite series through various examples.

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Magnetic fluid flow between two stationary parallel plates with constant pressure gradient

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Abstract: *In this paper we have found maximum velocity, average velocity and mass flow rate using ferrofluid with constant pressure gradient between stationary parallel plates.*

Keywords: *Ferrofluids, Couette flow, Maximum velocity, Average velocity, Mass flow rate.*

AMS Subject Classification: *76D08.*

1   Introduction

A ferrofluid or magnetic fluid is a liquid that become strongly magnetized in presence of a magnetic field. Ferrofluids are colloidal liquids made up of nano scale ferromagnetic or ferrimagnetic particles suspended in a carrier fluid. It is an artificial material rather than formed naturally. Ferrofluid is super-paramagnetic and create liquid seals held in position by magnetic fields [1].

Research is under way to create an adaptive optics shape-shifting magnetic mirror from magnetic fluid for Earth-based astronomical telescopes. Optical filters are used to select different wavelengths of light. The replacement of filters is cumbersome,especially when the wavelength is changed continuously with unable-type lasers. Optical filters tunable for different wavelengths by varying the magnetic field can be built using magnetic fluid emulsion.

Magnetic fluids enable an interesting opportunity to harvest vibration energy from the environment. Existing methods of harvesting low frequency ($< 100Hz$) vibrations require the use of solid resonant structure. With magnetic fluid based energy harvester designs no longer need solid structure. One example of magnetic fluid based energy harvesting is discussed in the journal article [2].

It is also used in electromagnetic ferrofluid-based energy harvester.For this a ferrofluid is placed inside a container that is wrapped with a coil of wire.The ferrofluid is then externally magnetized using a permanent magnet.When external vibrations cause the ferrofluid to slosh around in the container, there is a change in magnetic flux fields with respect to the coil of wire.Through Faraday's law of electromagnetic induction,voltage is induced in the coil of wire due to change in magnetic flux. An external magnetic field imposed on a magnetic fluid with varying susceptibility (because of temperature gradient)result in a non-uniform magnetic body force, which leads to a form of heat transfer called thermomagnetic convection. This form of heat transfer can be useful when conventional convection heat transfer is inadequate for example in miniature micro scale devices or under reduced gravity conditions [3].

Ferrofluids of suitable composition can exhibit extremely large enhancement in thermal conductivity. The large enhancement in k is due to the efficient transport of heat through percolating nanoparticle paths. Special magnetic nanofluids with tunable thermal conductivity

to viscosity ratio can be used as multi-functional 'smart material' that can remove heat and also arrest vibrations. Such fluids may find applications in micro-fluidic devices and micro-electromechanical systems (MEMS) [4].

Ferrofluid have numerous optical applications because of their refractive properties ; that is,each grain, a micro-magnet, reflects light. These applications include measuring specific viscosity of a liquid placed between a polarizer and an analyzer, illuminated by a helium -neon laser [5].

Recently, it has been demonstrated that magnetic fluids of suitable composition can exhibit extremely large enhancement in thermal conductivity (k) (i.e. 300 of the base fluid thermal conductivity). Studies confirmed that the large enhancement in k is due to the efficient transport of heat through percolating nanoparticle paths paths. Special magnetic nanofluids with tunable thermal conductivity to viscosity ratio can be used as multi-functional 'smart materials' that can remove heat and also arrest vibrations(damper). Such fluids offer exciting applications in micro-fluidic devices, micro and nano electromechanical system (MEMS and NEMS) and other nano technology based miniature devices.

Optical filters are used to select different wavelengths of lights. The replacement of filters is cumbersome, especially when the wavelength is changed continuously with tunable type of lasers. Optical filters, tunable for different wavelengths by varying the magnetic field can be built using ferrofluid emulsion [2].

Prakash et. al. [6] They examined the plane porous slider bearing and showed that the coefficient of friction increases even though the effect of porosity decreases the load capacity and friction.

2   Geometry and Governing equations

The flow between two infinite parallel flat plates separated by a fixed distance h apart, in which the plates are stationary and a pressure gradient is constant such a flow is called Plane Poiseuille Flow.

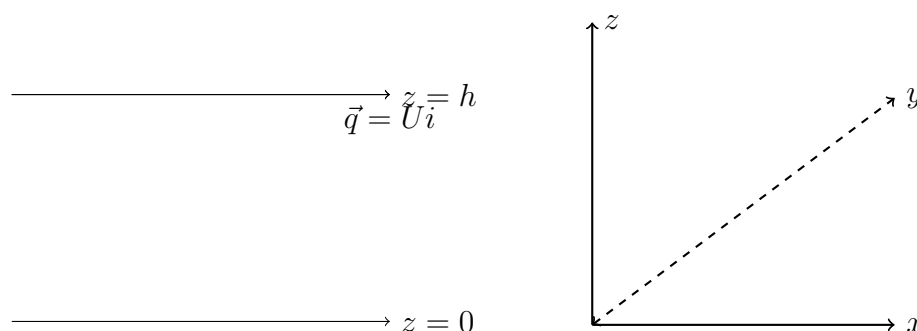Let x be the direction of the flow and z be the direction perpendicular to the flow.

Figure 1: Geometry of the flow

The velocity of the flow is given by

$$\vec{q} = u(x,y,z)\hat{i}, \ v = 0, \ w = 0, \tag{1}$$

From the continuity equation we have

$$\nabla \cdot \vec{q} = 0. \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{3}$$

since u is independent of x we have

$$\frac{\partial u}{\partial x} = 0. \tag{4}$$

The variation in the y direction are neglected since the flow is extending to infinity in the y direction. Therefore we have

$$\frac{\partial u}{\partial y} = 0, \tag{5}$$

hence we have

$$\vec{q} = u(z)\hat{i}. \tag{6}$$

The Navier-stokes equation [1, 3, 4] is given by

$$\rho \left[\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla)\vec{q} \right] = \rho \vec{g} - \nabla p + \mu_0 (\vec{M} \cdot \nabla)\vec{H} + \mu \nabla^2 \vec{q}. \tag{7}$$

Since the flow is steady we have

$$\frac{\partial \vec{q}}{\partial t} = 0. \tag{8}$$

Consider

$$(\vec{q} \cdot \nabla)\vec{q} = \left(u(z)\frac{\partial}{\partial x} \right) u(x)\hat{i} = 0, \tag{9}$$

Consider

$$\nabla^2 \vec{q} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \hat{i}, \tag{10}$$

$$\nabla^2 \vec{q} = \frac{\partial^2 u}{\partial z^2} = \frac{d^2 u}{dz^2}\hat{i}. \tag{11}$$

Let

$$\vec{M} = (M_x, M_y, M_z), \quad \vec{H} = (H_x, H_y, H_z), \tag{12}$$

consider

$$(\vec{M} \cdot \nabla)\vec{H} = \left[(M_x, M_y, M_z) \cdot \left(\frac{\partial}{\partial x}\hat{i}, \frac{\partial}{\partial y}\hat{j}, \frac{\partial}{\partial z}\hat{k} \right) \right] (H_x, H_y, H_z),$$
$$(\vec{M} \cdot \nabla)\vec{H} = (M_x \frac{\partial H_x}{\partial x} + M_z \frac{\partial H_x}{\partial z})\hat{i} + (M_x \frac{\partial H_z}{\partial x} + M_z \frac{\partial H_z}{\partial z})\hat{k}, \tag{13}$$

therefore equation (7) becomes

$$-\nabla p + \mu \frac{\partial^2 u}{\partial z^2}\hat{i} + \mu_0 \left(M_x \frac{\partial H_x}{\partial x} + M_z \frac{\partial H_x}{\partial z} \right) \hat{i} + \mu_0 \left(M_x \frac{\partial H_z}{\partial x} + M_z \frac{\partial H_z}{\partial z} \right) \hat{k} = 0. \tag{14}$$

In component form equations can be written as

$$\frac{\partial P}{\partial z} - \mu_0 \left(M_x \frac{\partial H_z}{\partial x} + M_z \frac{\partial H_z}{\partial z} \right) = 0, \tag{15}$$

$$-\frac{\partial P}{\partial z} + \mu_0 \left(M_x \frac{\partial H_z}{\partial x} + M_z \frac{\partial H_z}{\partial z} \right) = 0. \tag{16}$$

For irrotational flow [1] we have

$$\vec{H} = -\nabla\phi, \ \ \nabla \times \vec{H} = 0. \tag{17}$$

From Maxwell's equations [1, 4, 5, 7] we have

$$\nabla \cdot \vec{B} = 0,$$
$$\vec{B} = \mu_0(\vec{H} + \vec{M}). \tag{18}$$

The magnetization [3, 4] equation for ferrofluid is given by

$$\vec{M} = \overline{\mu}\vec{H}. \tag{19}$$

From equations (17)

$$H_x = -\frac{\partial \phi}{\partial x},$$
$$H_z = -\frac{\partial \phi}{\partial z}. \tag{20}$$

From equation (19) and (20)

$$M_x = \overline{\mu}H_x, \ \ M_z = \overline{\mu}H_z. \tag{21}$$

$$\vec{H}^2 = H_x^2 + H_z^2. \tag{22}$$

Consider

$$
\begin{aligned}
M_x \frac{\partial H_x}{\partial x} + M_z \frac{\partial H_x}{\partial z} &= \overline{\mu}H_x \frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial x}\right) + \overline{\mu}H_z \frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial x}\right), \\
&= \overline{\mu}\left(-\frac{\partial \phi}{\partial x}\right)\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial x}\right) + \overline{\mu}\left(-\frac{\partial \phi}{\partial z}\right)\frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial x}\right), \\
&= \frac{\overline{\mu}}{2}\frac{\partial}{\partial x}\left(H_x^2 + H_z^2\right), \\
M_x \frac{\partial H_x}{\partial x} + M_z \frac{\partial H_x}{\partial z} &= \frac{\overline{\mu}}{2}\frac{\partial}{\partial x}H^2.
\end{aligned}
\tag{23}
$$

Now consider

$$
\begin{aligned}
M_x \frac{\partial H_z}{\partial x} + M_z \frac{\partial H_z}{\partial z} &= \overline{\mu}H_x \frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial z}\right) + \overline{\mu}H_z \frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial z}\right), \\
&= \overline{\mu}\left(-\frac{\partial \phi}{\partial x}\right)\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial z}\right) + \overline{\mu}\left(-\frac{\partial \phi}{\partial z}\right)\frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial z}\right), \\
&= \frac{\overline{\mu}}{2}\frac{\partial}{\partial z}\left(H_x^2 + H_z^2\right), \\
M_x \frac{\partial H_z}{\partial x} + M_z \frac{\partial H_z}{\partial z} &= \frac{\overline{\mu}}{2}\frac{\partial}{\partial z}H^2.
\end{aligned}
\tag{24}
$$

Nomenclature: $\vec{q}$ is the velocity of the fluid, ρ is the density of the fluid, $\vec{g}$ is the gravitation, P is the pressure, $\vec{M}$ is the magnetization, $\vec{H}$ is the magnetic field intensity, μ_0 is the permeability of free space, $\overline{\mu}$ is the magnetic susceptibility, η is the coefficient of viscosity of the fluid.

3 Method of Solution

Substituting equations (23) and (24) in equations (15) and (16) we get

$$\mu\frac{\partial^2 u}{\partial z^2} - \frac{\partial}{\partial x}\left(p - \frac{\mu_0\overline{\mu}}{2}H^2\right) = 0, \tag{25}$$

$$\frac{\partial}{\partial z}\left(p - \frac{\mu_0\overline{\mu}}{2}H^2\right) = 0. \tag{26}$$

From equation (25) we have

$$\mu\frac{\partial^2 u}{\partial z^2} - A = 0, \tag{27}$$

where $A = \frac{\partial}{\partial x}\left(p - \frac{\mu_0\overline{\mu}}{2}H^2\right)$. Integrating the equation (27) we get

$$u = \frac{1}{\mu}\left(A\frac{z^2}{2} + cz + D\right), \tag{28}$$

where C, D are constants to be determined.

Boundary Conditions:
Since the fluid is viscus and the plates are stationary we have the boundary conditions

$$u = 0 \quad \text{at} \quad z = 0, \tag{29}$$

$$u = 0 \quad \text{at} \quad z = h. \tag{30}$$

Using the boundary conditions in equation (28) we get

$$\begin{aligned} D &= 0, \\ C &= -\frac{Ah}{2}. \end{aligned} \tag{31}$$

Substituting the values of C, D in equation (28) we get

$$\begin{aligned} u &= \frac{1}{\mu}\left(\frac{Az^2}{2} - \frac{Ah}{2}z\right), \\ \frac{u}{U} &= \frac{Ah^2}{2\mu U}\left(\frac{z^2}{h^2} - \frac{z}{h}\right), \\ u^* &= \frac{Ah^2}{2\mu U}\left(z^{2*} - z^*\right), \end{aligned} \tag{32}$$

where $u^* = \frac{u}{U}$ is the non-dimensional velocity and $z^* = \frac{z}{h}$ is non-dimensional vertical length.
Neglecting the asterisk $(*)$ for simplicity and substituting the value of A we get

$$u = -\frac{Ah^2}{2\mu}(z - z^2). \tag{33}$$

Therefore equation (33) is called velocity distribution of plane Poiseuille flow. Also equation (33) can be written as

$$u = \frac{G}{2}\left(z - z^2\right), \tag{34}$$

where $G = -\frac{Ah^2}{\mu U}$.

3.1   Maximum Velocity

Differentiate equation (33) with respect to z and equate it to zero we get

$$z = \frac{1}{2}, \tag{35}$$

also

$$\frac{\partial^2 u}{\partial z^2} = -G < 0, \tag{36}$$

therefore the maximum velocity is given by

$$u_{max} = -\frac{Ah^2}{8\mu U}, \tag{37}$$

$$u_{max} = -\frac{h^2}{8\mu U} \frac{\partial}{\partial x}\left(p - \frac{\mu_0 \overline{\mu}}{2} H^2\right). \tag{38}$$

Therefore equation (38) gives the maximum velocity.

3.2   Average Velocity

The average velocity is denoted by $\overline{u}$ and is defined as the ratio of total flow over a cross section to the area of the cross section.

$$\begin{aligned}
\overline{u} &= \frac{1}{h}\int_0^h u\,dz, \\
\overline{u} &= \frac{1}{h}\int_0^1 u^* U h\,dz^*, \\
\overline{u} &= -\frac{h^2}{12\mu U}\left\{\frac{\partial}{\partial x}\left(p - \frac{\mu_0 \overline{\mu}}{2} H^2\right)\right\},
\end{aligned} \tag{39}$$

and

$$\overline{u} = \frac{2}{3} u_{max}, \tag{40}$$

where $\overline{u^*} = \dfrac{\overline{u}}{U}$ is the non-dimensional average velocity.

3.3   Mass Flow Rate

The mass flow rate denoted by M and is defined as the amount of fluid that passes through any cross section of the channel per unit width and per unit time.

$$\begin{aligned}
M &= \rho\int_0^h u\,dz, \\
M &= -\frac{\rho h^3}{12 U \mu}\left\{\frac{\partial}{\partial x}\left(p - \frac{\mu_0 \overline{\mu}}{2} H^2\right)\right\},
\end{aligned} \tag{41}$$

therefore equation (41) gives the mass flow rate.

4   Results and Discussions

Average velocity and mass flow rate are determined for ferrofluid in the absence of pressure gradient. We found that the average velocity is constant, whereas mass flow rate is proportional to the density of the fluid.

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Finite Difference Method For Nonlinear Equations

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Abstract: *In this paper we will focus on the Finite difference method involved in solving systems of nonlinear boundary value problems for ordinary differential equations. Here, we have applied Successive iteration method and Newton method to find the unknowns which is later compared with the exact solution. We will give some applications as well as the advantages and disadvantages of Finite difference method. And we will solve few boundary value problem of a nonlinear ordinary differential equation using finite difference method.*

Keywords: *Finite Difference Method, Nonlinear Equations, Boundary value problems, Ordinary differential equations, Successive iteration method and Newton method.*

AMS Subject Classification: *65N06.*

1 Introduction

The finite difference approximations for the derivatives are one of the simplest and oldest methods to solve differential equations. It was already known by L. Euler (1707-1783) in the year 1768, in one dimension of space and was probably extended to dimension two by C. Runge (1856-1927) in the year 1908. The advent of finite difference techniques in numerical applications began in the early 1950s and their development was stimulated by the emergence of computers that offered a convenient framework for dealing with complex problems of science and technology.

The principle of finite difference methods is similar to the numerical schemes used to solve ordinary differential equations. The domain is partitioned in space and approximations of the solution are compute at the space. The error between the numerical solution and the exact solution is determined by the error that is committed by going from a differential operator to a difference operator. This error is called the discretization error or truncation error. The term truncation error reflects the fact that a finite part of a Taylor series is used in the approximation.

In Math 3351, Courtney Remani focused on solving nonlinear equations involving only a single variable. They used many methods like Newton's method, the Secant method, and the Bisection method and also they examined numerical methods such as Runge-Kutta method that are used to solve initial-value problems for Ordinary differential equations. They focused only on solving nonlinear equations with only one variable rather than nonlinear equations with several variable [1].

2 Finite difference method

Finite difference methods (FDM) are numerical methods for solving differential equations by approximating them with difference equations, in which finite differences approximate the

derivatives. Finite difference method convert a non-linear ordinary differential equations into a system of non-linear equations which can be solved by matrix algebra techniques. It computes the solutions numerically at a predefined set of discrete points in the structured grid of a computational domain. These discrete points along with their inter connections are called nodal points of the grid or mesh. The procedure of identifying the grid points for a given domain is called the discretization of the domain, which is the first step in the finite difference method [2].

The finite difference method approximates the differential operator by replacing the derivatives in the equation using differential quotients, which involve values of the solution at discrete mesh points in the domain under study. Repeated applications of this representation set up algebraic systems of equations in terms of unknown mesh point values. The method is a classical one, having been established almost a century ago. Timoshenko and Goodier (1970) provided some details on its applications in elasticity. The major difficulty with this scheme lies in the inaccuracies in dealing with regions of complex shape, although this problem can eliminated through the use of coordinate transformation techniques [3].

Weighted residual methods form a class of methods that can be used to solve differential equations. They make use of approximation functions that are appropriately weighted in order to find a solution which approximates the solution to the differential equations as closely as possible. Weighted residual methods are used in several other commonly encountered methods for solving differential equations numerically. It forms the basis for most of the numerical schemes.

The concept of FDM is focused on approximating differentials. In contrast to this, weighted residual methods evaluate the integral of differential equation and optimize an approximation such that the integrals of the correct and the approximated solutions match on a given domain. Therefore these equations use integral approximations. FDM uses an approximation of the differential of the differential equation. Hence it is a differential approximation. The mathematics of FDM is based on Taylor series approximations. The most common equations are

- Central finite difference scheme ((6)), for approximating first derivatives.
- Forward finite difference scheme ((4)), for approximating first derivatives.
- Backward finite difference scheme((5)), for approximating first derivatives.
- Central finite difference scheme ((7)), for approximating second derivatives.

These schemes are used in many forms in numerical solvers. The difference in the solution i.e., the finite change of the solution is approximated on a very small finite interval using one of these equations. All of these equations are linear, i.e., the solution is linearly approximated. Obviously, this approximation is only correct if the interval on which the function is linearized is sufficiently small. Otherwise, the solution becomes inexact [4].

The particular difference quotient and step size h are chosen to maintain a specified order of truncation error. However, h cannot be chosen too small because of the general instability of the derivative approximation. In the finite difference method, the approximation solutions are found by solving a set of algebraic equations that are the discrete representation of the governing differential equations and the boundary conditions. The discrete representation is formed by replacing the derivatives in the governing equations and the boundary conditions with approximations expressed in terms of difference between nodal displacements [5].

The finite difference method for the nonlinear equation requires the replacement of y'' and

y' by difference quotients, which results in a nonlinear system. This system can be solved using successive iterative method and Newton's method.

Newton's method, also known as the Newton-Raphson method, named after Isaac Newton and Joseph Raphson, is a root-finding algorithm which produces successively better approximations to the roots (or zeros) of a real-valued function. It is a powerful technique for solving equations numerically. It is based on the simple idea of linear approximation. The Newton method, properly used on a root with devastating efficiency. The Newton-Raphson method is widely used in finding the root of nonlinear equations. Newton's method converges quadratically. While carrying out this method the system converges quite rapidly once the approximation is close to the actual solution of the nonlinear system. This is seen as a advantage because Newton's method required less iterations, compared to another method with a lower rate of convergence, to reach the solution. However, when the system does not converge then an error in the computations occurs or a solution may not exist [6].

2.1   Advantages and Disadvantages of finite difference method

An important advantage of the finite difference method is its simplicity. Another advantage is the possibility to easily obtain high order approximations, and hence to achieve high order accuracy of the spatial discretization. On the other hand, because the method requires a structured grid, the range of application is clearly restricted. Furthermore, the finite difference method cannot be directly applied in body-fitted(curvilinear) coordinates, but the governing equation have to be first transformed into a Cartesian coordinate system. The problem herewith is that the Jacobian coordinate transformation appears in the flow equations. This Jacobian has to be discretized consistently in order to avoid the introduction of additional numerical errors.

Finite difference methods are the easiest numerical method to understand and implement differential equations, for problems that satisfy its structured discretization assumptions, and can be useful in other domain when we need to estimate other derivatives. It is most transparent and the most general method among the various numerical approaches. It has a straight forward nature and a minimum requirement on hardware.

The problem with Finite difference method is that in their most basic form, aren't applicable to unstructured domains. It is difficult to solve large, sparse system of matrices. Approximation property will ensure the error(difference between exact solution and finite difference method). They quickly become unwieldy if we need to start adding any sort of complexity like moving boundaries or an unstructured grid.

The FDM has better stability characteristics, but they generally requires more computation to a specified accuracy. The approximations may not be as accurate as the other numerical method for non-linear equation, there is less sensitivity to round off error.

2.2   Applications

1. Finite difference method's are very viable numerical methods for solution of partial differential equation and hence is suitable for solving plate binding equation. This method is sufficiently accurate for this thin plate analysis.

2. It is used in Power-flow problem formulation. Due to the nonlinear nature of this problem, numerical methods are employed to obtain a solution that is within an acceptable tolerance. The solution to the Power-flow problem begins with identifying the known and unknown variables in the system. Hence FDM's are used.

3. Finite difference methods (FDM) are used to numerically solve the elastodynamic wave equations. Finite difference techniques are applied to approximate both the time and space derivatives and are combined in various ways to provide different numerical algorithms for modeling elastic wave propagation.

4. The finite difference method is directly applicable only to rather simple geometries. Nowadays, it is utilized in the research of turbulent flows and together with immersed boundary cells in biology.

5. The FDM is a time-domain technique, which can find the concentration of dye everywhere in the computational domain at a given time frame. Burley et al., Wai and Vosoughi solved their convective dye transfer model equation by the FDM presented the results in the form of a number of graphs representing the variation in the concentration of dye at various points in the dyeing machine with time. Shannon et al. used the finite difference method to obtain the solution of their flow model equations, which predicts pressure and velocity profiles based on user defined package geometry, permeability profile and fluid properties [7].

2.3 Theorem

Suppose the function f in the boundary value problem,

$$y'' = f(x, y, y'), \ a \leq x \leq b, \ y(a) = \alpha \text{ and } y(b) = \beta$$

is continuous on the set,

$$D = ((x, y, y')|a \leq b, -\infty < y < \infty, -\infty < y' < \infty$$

and that the partial differential derivatives f_y and $f_{y'}$ are also continuous in D. If,

1. $f_y(x, y, y') > 0$ for all $(x, y, y') \in D$, and

2. a constant M exists with $|f_y(x, y, y')| < M$ for all $(x, y, y') \in D$, then the boundary value problem has a unique solution.

The numerical method we will be looking at is the finite difference method. This method can be used to solve both linear and nonlinear ordinary differential equations. Here we are considering the nonlinear finite difference method. Let nonlinear boundary value problem is of the form,

$$y'' = f(x, y, y'), \ a \leq x \leq b, \ y(a) = \alpha \text{ and } y(b) = \beta.$$

In order for the finite difference method to be carried out we have to assume f satisfies the following conditions,

1. f and the partial derivatives f_y and $f_{y'}$ are all continuous on

$$D = \{(x, y, y')|a \leq x \leq b, -\infty < y, y' < \infty\}$$

2. $f_y(x, y, y') \geq \delta$ on D, for some $\delta > 0$.

3. Constants k and L exists, with

$$k = max_{(x,y,y')\in D}|f_y(x, y, y')|, \text{ and } L = max_{(x,y,y')\in D}|f_{y'}(x, y, y')|[8].$$

3   Method of solution

Consider the nonlinear boundary value problems (BVPs) for the second order differential equation of the form,

$$y'' = f(x, y, y'), \ a \leq x \leq b, y(a) = \alpha \text{ and } y(b) = \beta. \tag{1}$$

Consider the finite-difference method for $y'(x)$ and $y''(x)$ be,

$$y'(x) = \frac{1}{2h}(y(x+h) - y(x-h)) - \frac{h^2}{6}y'''(x^{\star\star\star}) \tag{2}$$

$$y''(x) = \frac{1}{h^2}(y(x+h) - 2y(x) + y(x-h)) - \frac{h^2}{12}y'''(x^{\star\star\star}) \tag{3}$$

where $x^{\star\star\star}$ is between $x - h$ and $x + h$.
By neglecting the higher order terms in equation (2) we get,
Forward difference approximation is,

$$y' = \frac{y(x+h) - y(x)}{h} \tag{4}$$

Backward difference approximation is,

$$y' = \frac{y(x) - y(x+h)}{h} \tag{5}$$

Adding equation (4) and (5) we obtain,

$$y'' = \frac{y(x+h) - y(x-h)}{2h}. \tag{6}$$

Equation (6) is called central difference approximation for first order derivative.
By neglecting the higher order terms in equation (3) we get,

$$y'' = \frac{y(x+h) - 2y(x) + y(x-h)}{h^2} \tag{7}$$

Equation (7) is called central difference approximation for second order derivative.
Similar to the Finite difference method for linear boundary value problem we have,
$h = \dfrac{b-a}{N+1}$, and

$$x_0 = a$$
$$x_1 = a + h$$
$$\vdots$$
$$x_N = a + Nh$$
$$x_{N+1} = a + (N+1)h = b$$

Let, $y_0 = \alpha$, and $y_{N+1} = \beta$,
Apply central difference approximation formula to the equation(1) we get,

$$\frac{1}{h^2}(y_{i+1} - 2_i + y_{i-1}) = f\left(x_i, y_i, \frac{1}{2h}(y_{i+1} - y_{i-1})\right) \tag{8}$$

$$y_{i-1} - 2y_i + yi + 1 - h^2 f\left(x_i, y_i, \frac{1}{2h}(y_{i+1} - y_{i-1})\right) = 0$$

$$-y_{i-1} + 2y_i - yi + 1 + h^2 f(x_i, y_i, \frac{1}{2h}(y_{i+1} - y_{i-1}) = 0. \tag{9}$$

For $i = 1, 2, \cdots N$.

The $N \times N$ nonlinear system of equations obtained from this method is,

$$1 - y_2 + h^2 f\left(x_1, y_1, \frac{y_2 - \alpha}{2h}\right) - \alpha = 0,$$

$$-y_1 + 2y_2 - y_3 + h^2 f\left(x_2, y_2, \frac{y_3 - y_1}{2h}\right) = 0,$$

$$\vdots \qquad (10)$$

$$-y_{N-2} + 2y_{N-1} - y_N = h^2 f\left(x_{N-1}, y_{N-1}, \frac{y_N - y_{N-2}}{2h}\right) = 0,$$

$$-y_{N-1} + 2y_N + h^2 f\left((x_N, y_N, \frac{\beta - y_{N-1}}{2h}\right) - \beta = 0.$$

Equation (10) can be written in Matrix form as,

$$\text{diag}\{-1, 2, -1\} Y + h^2 F(x, Y) = AY + h^2 F(x, Y) = \overline{Y}. \qquad (11)$$

where,
$$Y = [y_1, \cdots y_N],$$
$$\overline{Y} = [\alpha,\ 0 \cdots\ 0,\ \beta],$$
$$A = \text{diag}\{-1, 2, -1\}$$

and

$$F(x, Y) = [F_1(x, Y)\ \cdots\ F_N(x, Y)], \quad \text{where} \quad F_i(x, Y) = f\left(x_i, y_i, \frac{1}{2h}(y_{i+1} - y_{i-1})\right).$$

$$(12)$$

We can find the initial approximation Y_k by the following equation,

$$Y_k = \alpha + \frac{\beta - \alpha}{b - a}(x_i - \alpha)$$

where, $x_i = \alpha + ih\ \forall\ i = 1, 2, \cdots N$.

It can be shown that the system of N nonlinear equations from (10) has a unique solution if $h < \dfrac{2}{L}$ where,

$$\left|\frac{\partial f(x, y, y')}{\partial y'}\right| \leq L \quad \text{for all } (x, y, y') \quad \text{in } D = \{(x, y, y') | a \leq x \leq b, -\infty < y, y' < \infty\}$$

Generally, the nonlinear equation in (10) cannot be solved exactly and then $y_1^{(k)} \cdots y_N^{(k)}$ are solved iteratively.

Let the initial values be $Y_0 = \left[y_1^{(0)} \cdots y_N^{(0)}\right]$, we have $Y_k = \left[y_1^{(k)} \cdots y_N^{(k)}\right]$ at the kth iteration and solve Y_{k+1} using following methods.

3.1   Successive iteration method

Y_{k+1} is the solution of the following system of linear equations,

$$AY_{k+1} + h^2 F(x, Y_k) = \overline{Y}$$
$$AY_{k+1} = \overline{Y} - h^2 F(x, Y_k)$$

3.2 Newton method

Let $G(x, Y) = AY + h^2 F(x, Y) - \overline{Y} = [g_1(x, Y) \cdots g_N(x, Y)]$.
The linearization of $G(x, Y)$ at Y_k is,

$$G(x, Y) \approx G(x, Y_k) + J(x, Y_k)(Y_{k+1} - Y_k).$$

where,

$$J(x, Y) = \left[\frac{\partial g_i(x, Y)}{\partial y_j}\right] = \begin{bmatrix} \dfrac{\partial g_1}{\partial y_1} & \dfrac{\partial g_1}{\partial y_2} & \cdots & \dfrac{\partial g_1}{\partial y_N} \\ \dfrac{\partial g_2}{\partial y_1} & \dfrac{\partial g_2}{\partial y_2} & \cdots & \dfrac{\partial g_2}{\partial y_N} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial g_N}{\partial y_1} & \dfrac{\partial g_N}{\partial y_2} & \cdots & \dfrac{\partial g_N}{\partial y_N} \end{bmatrix}$$

is the Jacobi Matrix of $G(x, Y)$. Then solve the system of linear equations for Y_{k+1}.

$$G(x, Y_k) + J(x, Y_k)(Y_{k+1} - Y_k) = 0$$
$$Y_{k+1} = Y_k - [J(x, Y_k)]^{-1} G(x, Y_k).$$

where,

$$J(x, Y) = A + h^2 J_F(x, Y)$$
$$J_F(x, Y) = \left[\frac{\partial F_i(x, Y)}{\partial y_j}\right]$$
$$J_F(x, Y) = \text{diag}\left\{\frac{\partial F_i(x, Y)}{\partial y_{j-1}}, \frac{\partial F_i(x, Y)}{\partial y_j}, \frac{\partial F_i(x, Y)}{\partial y_{j+1}}\right\}$$

In finite difference method, $J(y_1, y_2, \cdots, y_N)$ is tridiagonal with ij^{th} entry. This means that there are non-zero entries on the diagonal below the main diagonal, and there are non-zero entries on the diagonal directly above the main diagonal [8].

4 Examples and Discussions

Example 1: Use the nonlinear finite difference method with h=0.25 to approximate the solution to the boundary-value problem,

$$y'' = 2y^3, \ -1 \le x \le 0, \ y(-1) = \frac{1}{2}, \ y(0) = \frac{1}{3}.$$

Compare the results to the actual solution $y(x) = \dfrac{1}{x + 3}$.

Solution: Given,

$$y'' = 2y^3, \ -1 \le x \le 0, \ y(-1) = \frac{1}{2}, \ y(0) = \frac{1}{3}$$

Let, $\overline{x} = [-1, -0.75, -0.5, -0.25, 0] \quad \text{and} \quad \overline{Y} = \begin{bmatrix} \dfrac{1}{2} \\ 0 \\ \dfrac{1}{3} \end{bmatrix}$

$$Y_k = \alpha + \frac{\beta - \alpha}{b - a}(x_i - a) = \begin{bmatrix} 0.83333 \\ 0.16666 \\ 0.25 \end{bmatrix}$$

$$F(x, Y_k) = \begin{bmatrix} f\left(x_1, y_1, \left(\frac{1}{2h}(y_2 - \alpha)\right)\right) \\ f\left(x_2, y_2, \left(\frac{1}{2h}(y_3 - y_1)\right)\right) \\ f\left(x_3, y_3, \left(\frac{1}{2h}(\beta - y_3)\right)\right) \end{bmatrix} = 2 \begin{bmatrix} y_1^3 \\ y_2^3 \\ y_3^3 \end{bmatrix} \text{ and } F(x, Y_k) = \begin{bmatrix} (\frac{1}{12})^3 \\ (\frac{1}{6})^3 \\ (\frac{1}{4})^3 \end{bmatrix}$$

1. Successive iteration method

$$AY_{k+1} = \overline{Y} - h^2 F(x, Y_k)$$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{3} \end{bmatrix} - \frac{1}{16} \begin{bmatrix} \left(\frac{1}{12}\right)^3 \\ \left(\frac{1}{6}\right)^3 \\ \left(\frac{1}{4}\right)^3 \end{bmatrix}$$

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 0.4519 \\ 0.4158 \\ 0.3741 \end{bmatrix}$$

2. Newton method

$$Y_{k+1} = Y_k - [J(x, Y_k)]^{-1} G(x, Y_k)$$
$$G(x, Y_k) = AY_k + h^2 F(x, Y_k) - \overline{Y}$$

Let, $f(x, y, y') = 2y^3$ hence $F(x, Y_k) = \begin{bmatrix} 2y_1^3 \\ 2y_2^3 \\ 2y_3^3 \end{bmatrix}$ and $J(x, Y_k) = \begin{bmatrix} \frac{1}{48} & 0 & 0 \\ 0 & \frac{1}{12} & 0 \\ 0 & 0 & \frac{3}{16} \end{bmatrix}$

$$B = A + h^2 J(x, Y_k)$$
$$B = \begin{bmatrix} 2.0026 & -1 & 0 \\ -1 & 2.0104 & -1 \\ 0 & -1 & 2.0234 \end{bmatrix}$$
$$b = AY_k + h^2 F(x, Y_k) - \overline{Y}$$
$$b = \begin{bmatrix} -0.4999 \\ 0.00028 \\ 0.00094 \end{bmatrix}$$
$$Y_{k+1} = Y_k - B^{-1} b$$
$$Y_{k+1} = \begin{bmatrix} 0.4551 \\ 0.4114 \\ 0.3705 \end{bmatrix}$$

Exact Solution		Numerical solution	
		Successive iteration	Newton method
1	0.44444	0.4519	0.4551
2	0.4	0.4158	0.3705
3	0.3636	0.3741	0.3705

Example 2: Use the nonlinear finite difference method with h=0.25 to approximate the solution to the boundary-value problem,

$$y'' = -e^{-2y}, 1 \leq x \leq 2, y(1) = 0, y(2) = \ln(2).$$

Compare the results to the actual solution $\ln x$.

Solution: Given,

$$y'' = -e^{-2y}, 1 \leq x \leq 2, y(1) = 0, y(2) = \ln(2)$$

Let, $\overline{x} = [-1, -0.75, -0.5, -0.25, 0]$ and $\overline{Y} = [0, 0, 0.693147]$

$$Y_k = \alpha + \frac{\beta - \alpha}{b - a}(x_i - a) = \begin{bmatrix} 0.173286 \\ 0.3465735 \\ 0.511986025 \end{bmatrix}$$

$$F(x, Y_k) = \begin{bmatrix} f\left(x_1, y_1, \left(\frac{1}{2h}(y_2 - \alpha)\right)\right) \\ f\left(x_2, y_2, \left(\frac{1}{2h}(y_3 - y_1)\right)\right) \\ f\left(x_3, y_3, \left(\frac{1}{2h}(\beta - y_3)\right)\right) \end{bmatrix} = \begin{bmatrix} e^{-2y_1} \\ e^{-2y_2} \\ e^{-2y_3} \end{bmatrix} = \begin{bmatrix} -0.707107905 \\ -0.50000009 \\ -0.353553486 \end{bmatrix}$$

1. Successive iteration method

$$AY_{k+1} = \overline{Y} - h^2 F(x, Y_k) \tag{13}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.693147 \end{bmatrix} - \frac{1}{16} \begin{bmatrix} -0.707107905 \\ -0.50000009 \\ -0.353553486 \end{bmatrix} \tag{14}$$

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 0.2275817 \\ 0.4109692 \\ 0.5631066 \end{bmatrix} \tag{15}$$

2. Newton method

$$Y_{k+1} = Y_k - [J(x, Y_k)]^{-1} G(x, Y_k)$$
$$G(x, Y_k) = AY_k + h^2 F(x, Y_k) - \overline{Y}$$

Let, $f(x, y, y') = 2y^3$.

Hence $F(x, Y_k) = \begin{bmatrix} -0.707107905 \\ -0.50000009 \\ -0.353553486 \end{bmatrix}$ and $J(x, Y_k) = \begin{bmatrix} 1.4142113, 0, 0 \\ 0, 1.999999, 0 \\ 0, 02.828426 \end{bmatrix}$

$$B = A + h^2 J(x, Y_k)$$

$$B = \begin{bmatrix} 1.9116118 & -1 & 0 \\ -1 & 1.87500001 & \\ 0 & -1 & 1.8232234 \end{bmatrix}$$

$$b = AY_k + h^2 F(x, Y_k) - \overline{Y}$$

$$b = \begin{bmatrix} -0.0441957 \\ -0.0312493 \\ -0.0220971 \end{bmatrix}$$

$$Y_{k+1} = Y_k - B^{-1} b$$

$$Y_{k+1} = \begin{bmatrix} 0.2396983 \\ 0.4293324 \\ 0.5773716 \end{bmatrix}$$

	Exact Solution	Numerical solution	
		Successive iteration	Newton method
1	0.2231435	0.2275817	0.2396983
2	0.4054651	0.4109692	0.4293324
3	0.5596157	0.5631066	0.5773716

Example 3: Use the nonlinear finite difference method with h=0.25 to approximate the solution to the boundary-value problem,

$$y'' = y' + 2(y - \ln x)^3 - x^{-1}, \; 2 \le x \le 3, \; y(2) = \frac{1}{2} + \ln 2, \; y(3) = \frac{1}{3} + \ln 3.$$

Compare the results to the actual solution $x^{-1} + \ln x$.
Solution: Given,

$$y'' = y' + 2(y - \ln x)^3 - x^{-1}, \; 2 \le x \le 3, \; y(2) = \frac{1}{2} + \ln 2, \; y(3) = \frac{1}{3} + \ln 3$$

Let, $\overline{x} = \left[2, \frac{9}{4}, \frac{5}{2}, \frac{11}{4} \right], \quad \overline{Y} = \left[\frac{1}{2} + \ln 2, 0, \frac{1}{3} + \ln 3 \right]$

$$Y_k = \alpha + \frac{\beta - \alpha}{b - a}(x_i - a) = \begin{bmatrix} 0.3579 \\ 0.71595 \\ 1.07392 \end{bmatrix}$$

$$F(x, Y_k) = \begin{bmatrix} f\left(x_1, y_1, \left(\frac{1}{2h}(y_2 - \alpha) \right) \right) \\ f\left(x_2, y_2, \left(\frac{1}{2h}(y_3 - y_1) \right) \right) \\ f\left(x_3, y_3, \left(\frac{1}{2h}(\beta - y_3) \right) \right) \end{bmatrix} = \begin{bmatrix} -1.584781 \\ 1.01606208 \\ 1.06882890 \end{bmatrix}$$

1. Successive iteration method

$$AY_{k+1} = \overline{Y} - h^2 F(x, Y_k)$$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \ln 2 \\ 0 \\ \frac{1}{3} + \ln 3 \end{bmatrix} - \frac{1}{16} \begin{bmatrix} -1.584781 \\ 1.01606208 \\ 1.06882890 \end{bmatrix}$$

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 1.2786342 \\ 1.265111089 \\ 1.3151089 \end{bmatrix}$$

2. Newton method

$$Y_{k+1} = Y_k - [J(x, Y_k)]^{-1} G(x, Y_k)$$
$$G(x, Y_k) = AY_k + h^2 F(x, Y_k) - \overline{Y}$$

Let, $f(x, y, y') = y' + 2(y - \ln x)^3 - x^{-1}$.

Hence $F(x, Y_k) = \begin{bmatrix} 2(y_2 - y_0) + 2(y_1 - \ln x_1)^3 - x_1^{-1} \\ 2(y_3 - y_1) + 2(y_2 - \ln x_2)^3 - x_2^{-1} \\ 2(y_4 - y_0) + 2(y_3 - \ln x_3)^3 - x_3^{-1} \end{bmatrix}$ and

$$J(x, Y_k) = \begin{bmatrix} 0.05412 & 2 & 0 \\ -2 & 0.240818 & 2 \\ 0 & -2 & 0.02330 \end{bmatrix}.$$

$$B = A + h^2 J(x, Y_k)$$
$$B = \begin{bmatrix} 2.0769639 & -0.875 & 0 \\ -1.125 & 2.0150511 & -0.875 \\ 0 & -1.125 & 2.0014562 \end{bmatrix}$$
$$b = AY_k + h^2 F(x, Y_k) - \overline{Y}$$
$$b = \begin{bmatrix} -1.2922988 \\ 0.0635839 \\ 0.0691702 \end{bmatrix}$$
$$Y_{k+1} = Y_k - B^{-1} b$$
$$Y_{k+1} = \begin{bmatrix} 1.2234811 \\ 1.2936432 \\ 1.3640761 \end{bmatrix}$$

	Exact Solution	Numerical solution	
		Successive iteration	Newton method
1	1.25537	1.2786342	1.2234811
2	1.31629	1.2651196	1.2936432
3	1.37523	1.3151089	1.3640761

5   Conclusion

In this paper we studied about Finite difference method which is a powerful method in not only solving nonlinear algebraic equations with one variable, but also systems of nonlinear algebraic equations. Finite difference methods are also influential in solving for boundary value problems of nonlinear ordinary differential equations. In finite difference method we can solve the nonlinear system by three methods they are successive iteration method, Newton's method and Crout factorization algorithm. Here, the finite difference method implements both Successive iteration method and Newton's method once the boundary value problem was converted into a nonlinear algebraic system of equations. By the above solved problems we can conclude

that successive iteration method provide highly accurate values in less number of iteration as compared with Newton's method. Several numerical examples are solved to illustrate the efficiency and the performance of the finite difference method. It has better stability than shooting methods for boundary value problems. Higher-order differences or extrapolation can be used to improve accuracy. Finite difference method tend to less sensitive to round off error than shooting method.

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A study on various classification and clustering data mining algorithms

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Abstract: *Data mining is the process of extracting information from large database or datasets. Mining can be done by using supervised and unsupervised learning. Classification or the supervised learning is a two step process. In the first step, training data are analyzed by a classification algorithm. In the second step, test data are used to estimate the accuracy of the classification rules. If the accuracy is considered acceptable, the rules can be applied to the classification of new data tuples. The learning of classifier is supervised in that it is told to which class each training tuple belongs. It contrasts with unsupervised learning or clustering, in which the class label of each training tuple is not known in advance. The process of grouping a set of physical or abstract objects into classes of similar objects is called clustering. A cluster is a collection of data objects that are similar to one another within the same cluster and are dissimilar to the objects in other clusters. Clustering algorithms can be classified into partition-based algorithms, hierarchical based algorithms, density-based algorithms and grid-based algorithms. In this paper we will study about various data mining classification algorithms like K-nearest Neighbor, Naive Bayes, Support Vector Machine etc and clustering algorithms like K-Means, Hierarchical clustering etc.*

Keywords: *Data mining, Classification algorithms, Naive bayes, KNN, SVM, C4.5, K-Means, Hierarchical clustering, Decision tree.*

1   Introduction

Data Mining (also known as KDD) is defined as the automatic extraction of unidentified, useful and understandable patterns from large databases [1]. Data Mining has three major components, Clustering or Classification, Association Rules and Sequence Analysis. In classification/clustering analyze a set of data and generate a set of grouping rules which can be used to classify future data.

Classification-rule learning involves finding rules that partition given data into predefined classes. An association rule is a rule which implies certain association relationships among a set of objects in a database. Mining association rules may require iterative scanning of large transaction or relational databases which is quite costly in processing. In sequential Analysis, we seek to discover patterns that occur in sequence.

2   Classification Algorithms

Classification is used to classify each item in a set of data into one of predefined set of classes or groups. In the Web domain, Web master or marketer will have to use this technique if he/she want to establish a profile of users belonging to a particular class or category. This requires extraction and selection of features that best describe the properties of a given class or

category [2].

Classification algorithm assigns each instance to a particular class such that classification error will be least. It is used to extract models that accurately define important data classes within the given dataset. The classification can be done by using supervised inductive learning algorithms such as decision tree classifiers, Naive Bayesian classifiers, k-nearest neighbor classifier, Support Vector Machines etc.

2.1   K- nearest neighbor

KNN is a non-parametric supervised learning technique in which we try to classify the data point to a given category with the help of training set.

Let (X_i, C_i) where $i = 1, 2, \ldots n$ be data points. X_i denotes feature values and C_i denotes labels for X_i for each i. Assuming the number of classes as 'c', $C_i \in 1, 2, 3, \ldots c$ for all values of i. Let x be a point for which label is not known, and we would like to find the label class using k-nearest neighbor algorithms.

KNN Algorithm Pseudo code:
1. Calculate "$d(x, x_i)$" $i = 1, 2, \ldots n$, where d denotes the Euclidean distance between the points.
2. Arrange the calculated n Euclidean distances in non-decreasing order.
3. Let k be a positive integer, take the first k distances from this sorted list (closest distances).
4. Find those $k-$ points corresponding to these $k-$ distances.
5. Let k_i denote the number of points belonging to the ith class among k points i.e. $k \geq 0$.
6. If $k_i > k_j \forall i \neq j$ then put x in class j.
Time Complexity: $O(nk + nd)$

2.2   Naive Bayes

Naive Bayes Classifier is the simple Statistical Bayesian Classifier [3]. It is called Naive as it assumes that all variables contribute towards classification and are mutually correlated. This assumption is called class conditional independence. They can predict class membership probabilities, such as the probability that a given data item belongs to a particular class label. A Naive Bayes classifier considers that the presence (or absence) of a particular feature(attribute) of a class is unrelated to the presence (or absence) of any other feature when the class variable is given. The Naive Bayes Classifier technique is based on Bayesian Theorem and it is used when the dimensionality of the inputs is high. Bayes Theorem is stated as below:

Let X is a data sample whose class label is not known and let H be some hypothesis, such that the data sample X may belong to a specified class C. Bayes theorem is used for calculating the posterior probability $P(C|X)$ from $P(C), P(X)$, and $P(X|C)$.

$$P(c|x) = \frac{P(x|c)P(c)}{p(x)}$$

where $P(c|x)$ is Posterior probability, $P(x|c)$ is Likelihood, $P(c)$ is Class prior probability and $P(x)$ Predictor prior probability. Also, $P(c|X) = P(x_1|c) \times P(x_2|c) \times \ldots \times P(x_n|c) \times P(c)$.

The Naive Bayes classifier [3] works as follows:

1. Let D be the training dataset associated with class labels. Each tuple is represented by $n-$dimensional element vector, $X = (x_1, x_2, x_3, \ldots, x_n)$.

2. Consider that there are m classes $C_1, C_2, C_3, \ldots, C_m$. Suppose that we want to classify an unknown tuple X, then the classifier will predict that X belongs to the class with higher posterior probability, conditioned on X. i.e., the Naive Bayesian classifier assigns an unknown tuple X to the class C_i if and only if $P(C_i|X) > P(C_j|X)$ for $1 \le j \le m$ and $i \neq j$, above posterior probabilities are computed using Bayes Theorem.

Time Complexity: $O(Np)$, where N is the number of training examples and p is the number of features.

2.3   Support vector machines (SVM)

Support Vector Machine (SVM) [4] is primarily a classier method that performs classification tasks by constructing hyperplanes in a multidimensional space that separates cases of different class labels. SVM supports both regression and classification tasks and can handle multiple continuous and categorical variables. For categorical variables a dummy variable is created with case values as either 0 or 1. Thus, a categorical dependent variable consisting of three levels, say (A, B, C), is represented by a set of three dummy variables: $A : (100), B : (010), C : (001)$.

To construct an optimal hyperplane, SVM employs an iterative training algorithm, which is used to minimize an error function.

In a two-class learning task, the aim of SVM is to find the best classification function to distinguish between members of the two classes in the training data. The metric for the concept of the "best" classification function can be realized geometrically. For a linearly separable dataset, a linear classification function corresponds to a separating hyperplane $f(x)$ that passes through the middle of the two classes, separating the two. Once this function is determined, new data instance x_n can be classified by simply testing the sign of the function $f(x_n)$; x_n belongs to the positive class if $f(x_n) > 0$. Because there are many such linear hyperplanes, what SVM additionally guarantee is that the best such function is found by maximizing the margin between the two classes. Intuitively, the margin is defined as the amount of space, or separation between the two classes as defined by the hyperplane. Geometrically, the margin corresponds to the shortest distance between the closest data points to a point on the hyperplane. Having this geometric definition allows us to explore how to maximize the margin, so that even though there are an infinite number of hyperplanes, only a few qualify as the solution to SVM. The reason why SVM insists on finding the maximum margin hyperplanes is that it offers the best generalization ability. It allows not only the best classification performance (e.g. accuracy) on the training data, but also leaves much room for the correct classification of the future data. To ensure that the maximum margin hyperplanes are actually found, an SVM classifier attempts to maximize the following function with respect to w, b

$$L_p = \frac{1}{2}\|w\|^2 + C\sum_{i=1}^{n}\xi_i - \sum_{i=1}^{n}\alpha_i\{y_i(x_i^T w + b) - 1 + \xi_i\} - \sum_{i=1}^{n}\mu_i\xi_i$$

where n is the number of training examples and $\alpha_{i,i}, i = 1, \ldots, t$ are non-negative numbers such that the derivatives of LP with respect to α_i are zero. α_i are the Lagrange multipliers and

LP is called the Lagrangian. In this equation, the vectors w and constant b define the hyperplane.

Time Complexity: $O(N3)$,where n is the number of free support vectors.

2.4   Decision Tree Induction

A decision tree is a flow chart like tree structure, where each node denotes test on an attribute value, each branch represents the result of the test, and tree leaves represent classes [5]

General idea: Recursively partition data into sub-groups
- ⋆ Select an attribute and formulate a logical test on attribute
- ⋆ Branch on each outcome of test, move subset of examples (training data) satisfying that outcome to the corresponding child node.
- ⋆ Run recursively on each child node.
- ⋆ Termination rule specifies when to declare a leaf node.
- ⋆ If all instances are correctly classified - stop.
- ⋆ If an instances is incorrectly classified, add it to the initial subset and construct a new tree.
- ⋆ Iterate until, A tree is built that classifies all instance correctly OR A tree is built from the entire training set
- ⋆ Decision tree learning is a heuristic, one-step look ahead (hill climbing), non-backtracking search through the space of all possible decision trees. Decision trees are simple to understand and provide good results even with small data. Decision tree induction algorithms can be used for classification in many application areas, such as Education, Medicine, Manufacturing, Production, Financial analysis, Fraud Detection and Astronomy etc. There are several data mining algorithms such as C4.5, ID3, CART, J48, NB Tree, REP Tree etc.

Complexity Building a tree $O(mn \log n)$
Sub tree replacement $O(n)$
Sub tree raising $O(n(n \log n)^2$
Every instance may have to be redistributed at every node between its leaf and the root $O(n \log n)$
Cost for redistribution(on average) $O(\log n)$
Total Cost $O(mn \log n) + O(n(n \log n)^2$. [6]

2.5   C4.5

C4.5 is an algorithm used to generate a decision tree developed by Ross Quinlan [7]. C4.5 is an extension of Quinlan's earlier ID3 algorithm. C4.5 is implemented recursively with this following sequence
- (i) Check if algorithm satisfies termination criteria.
- (ii) Computer information-theoretic criteria for all attributes.
- (iii) Choose best attribute according to the information-theoretic criteria.
- (iv) Create a decision node based on the best attribute in step (iii).
- (v) Induce (i.e. split) the dataset based on newly created decision node in step (iv).
- (vi) For all sub-dataset in step (v), call C4.5 algorithm to get a sub-tree (recursive call).
- (vii) Attach the tree obtained in step (vi) to the decision node in step (iv).
- (viii) Return tree.

3   Clustering Algorithms

Clustering is a common technique for statistical data analysis, which is used in many fields. Clustering can be classified into the following categories: Partitioning clustering, Hierarchical clustering , Density based clustering, Model based clustering, Grid based clustering. These methods vary in

⋆ the procedures used for measuring the similarity (within and between clusters)

⋆ the use of thresholds in constructing clusters

⋆ the manner of clustering, that is, whether they allow objects to belong to strictly to one cluster or can belong to more clusters in different degrees and the structure of the algorithm. Irrespective of the method used, the resulting cluster structure is used as a result in itself, for inspection by a user, or to support retrieval of objects [8].

Partitioning Algorithms

Partitioning Clustering split the data into k divisions ,where each division represent a cluster, $k \leq n$, where n is the number of data points. Partitioning methods are based on the idea that a cluster is represented by a centre point. The cluster must exhibit the following two properties

⋆ each collection should have at least one object

⋆ every object should belong to accurately one collection.

The drawback of this algorithm is whenever a point is close to the center of another cluster, it gives poor outcome due to overlapping of data points. It uses number of greedy heuristics of iterative optimization.

Partitioning cluster methods are: K-Means, K-Mediods method, PAM (Partitioning around Mediods), CLARA (Clustering Large Applications).

3.1   K-Means Algorithm

It is a centroid based technique. This algorithm takes the input parameters k and partition a set of n objects into k clusters that the resulting intra-cluster similarity is high but the inter-cluster similarity is low. The method can be used by cluster to assign rank values to the cluster categorical data is statistical method. k mean is mainly based on the distance between the object and the cluster mean. Then it computes the new mean for each cluster. Here categorical data have been converted into numeric by assigning rank value [9].

Algorithm: In this we take k the number of cluster and D as data set containing an object. In this output is stored as a set of k clusters. Algorithm follows some steps these are

Step 1 Randomly choose k object from D as initial cluster center.

Step 2 Calculate the distance from the data point to each cluster.

Step 3 If the data point is closest to its own cluster, leave it where it is. If the data point is not closest to its own cluster, move it into the closest cluster.

Step 4 Repeat step 2 and 3 until best relevant cluster is found for each data.

Step 5 Updates the cluster means and calculate the mean value of the object for each cluster.

Step 6 Stop (every data is located in a proper positioned cluster)

Time Complexity: $O(n2)$, where n is the input data size.

3.2   Hierarchical Clustering

Hierarchical algorithms find successive clusters using previously established clusters, whereas partitional algorithms determine all clusters at a time. Hierarchical algorithms can be agglomerative (bottom-up) or divisive (top-down). Agglomerative algorithms begin with each element as a separate cluster and merge them in successively larger clusters. Divisive algorithms begin with the whole set and proceed to divide it into successively smaller clusters. The decision of merging two clusters is taken on the basis of closeness of these clusters.

Time Complexity: $O(n^2 \log n)$

There are multiple metrics for deciding the closeness of two clusters :
* Euclidean distance: $\|a - b\|2 = \sqrt{\Sigma(a_i - b_i)}$
* Squared Euclidean distance: $\|a - b\|22 = \sqrt{\Sigma(a_i - b_i)^2}$
* Manhattan distance: $\|a - b\|1 = \Sigma(a_i - b_i)$
* Maximum distance: $\|a - b\| \text{ INFINITY} = \max_i \|a_i - b_i\|$
* Mahalanobis distance: $\sqrt{(a - b)TS - 1(-b)}$ where S is a covariance matrix

3.3   Agglomerative Hierarchical clustering

This algorithm works by grouping the data one by one on the basis of the nearest distance measure of all the pairwise distance between the data point. Again distance between the data point is recalculated but which distance to consider when the groups has been formed. For this there are many available methods. Some of them are:
1. Single-nearest distance or single linkage.
2. Complete-farthest distance or complete linkage.
3. Average-average distance or average linkage.
4. Centroid distance.
5. Ward's method - sum of squared Euclidean distance is minimized.

This way we go on grouping the data until one cluster is formed. Now on the basis of dendogram graph we can calculate how many number of clusters should be actually present.

Algorithmic steps for Agglomerative Hierarchical clustering
Let $X = \{x_1, x_2, x_3, \ldots, x_n\}$ be the set of data points.
  i) Begin with the disjoint clustering having level $L(0) = 0$ and sequence number $m = 0$.
 ii) Find the least distance pair of clusters in the current clustering, say pair $(r), (s)$, according to $d[(r), (s)] = \min d[(i), (j)]$ where the minimum is over all pairs of clusters in the current clustering.
iii) Increment the sequence number: $m = m + 1$. Merge clusters (r) and (s) into a single cluster to form the next clustering m. Set the level of this clustering to $L(m) = d[(r), (s)]$.
iv) Update the distance matrix, D, by deleting the rows and columns corresponding to clusters (r) and (s) and adding a row and column corresponding to the newly formed cluster. The distance between the new cluster, denoted (r, s) and old cluster(k) is defined in this way: $d[(k), (r, s)] = \min (d[(k), (r)], d[(k), (s)])$.
 v) If all the data points are in one cluster then stop, else repeat from step ii).

Divisive Hierarchical clustering: It is just the reverse of Agglomerative Hierarchical approach.
Time Complexity: $O(n^3)$

Table 1: Comparison of classification algorithms

Algorithm	Pros	Cons
Decision Tree [10]	⋆ It is easy to understand. ⋆ It provides fast result in classifying unknown records. ⋆ It provides good results with small size tree. Results does not affect with outliers. ⋆ It does not require preparation method like normalization. ⋆ It also works well with numeric data.	⋆ It can't predict the value of a continuous class attribute. ⋆ It provides error prone results when too many classes are used. ⋆ Irrelevant attribute affect construction of decision tree in a bad manner. ⋆ Small change in data can change the decision tree completely. ⋆ It is sensitive to noise.
C4.5	⋆ It uses both continuous and discrete data. ⋆ It avoids over fitting of data. ⋆ It improves computational efficiency,easy to implement. ⋆ Deals with noise. ⋆ Build models can be easily interpreted. ⋆ It handles training data with missing and numeric value.	⋆ It requires that target attribute will have only discrete values. ⋆ Overfitting ⋆ Does not work very well on a small training dataset. ⋆ Small variation in data can lead to different decision trees.
SVM [5]	⋆ High accuracy ⋆ Work well even if data is not linearly separable in the base feature space.	⋆ Speed and size requirement both in training and testing is more. ⋆ It is sensitive to noise. ⋆ High complexity and extensive memory requirements for classification in many cases.
KNN	⋆ It performs better with missing data. ⋆ It is easy to implement and debug. ⋆ It provides more accurate results. ⋆ Some noise reduction techniques are used that improve the accuracy of classifier. ⋆ Classes need to be linearly separable. ⋆ Zero cost of the learning process. ⋆ Sometimes it is robust with regard to noisy training data. ⋆ Well suited for multimodal classes.	⋆ It has poor run time performance time to find the nearest neighbours in a large training set can be excessive ⋆ It requires high calculation complexity. ⋆ It considers no weight difference between samples. ⋆ It is sensitive to irrelevant attributes . ⋆ Performance of algorithm depends on the number of dimensions used.

Naive Bayesian	⋆ It provides high accuracy and speed on large database. ⋆ It has minimum error rate in comparison to all other classifiers. ⋆ It is easy to understand, implement. ⋆ It is not sensitive to irrelevant features. ⋆ It handles streaming data well. ⋆ It can also handle real and discrete values. ⋆ Great computational efficiency and classification rate. ⋆ When assumption of independence holds, a Naive Bayes classifier performs better compare to other models.	⋆ If categorical variable has a category (in test data set), which was not observed in training data set, then model will assign a zero probability and will be unable to make prediction. This is often known as "Zero frequency". To solve this, we can use simplest smoothing techniques like Laplace estimation. ⋆ It assumes independence of features. So it provides less accuracy in case of independent predictors. ⋆ The precision of algorithm decreases if the amount of data is less.

Table 2: Comparison of Clustering Algorithms

| Algorithm	Pros	Cons
K-means	⋆ Feasible and Scalable [11] ⋆ KNN is well suited for multi-modal classes as well as applications in which an object can have many class labels [4] ⋆ Its effective in dealing with large data sets. ⋆ Works well with clusters having convex shapes. ⋆ It frequently terminates at local optimum. ⋆ Gives best result when data set are distinct or well separated from each other.	⋆ Sensitive to initial parameter K (number of clusters). It can only be used when mean is defined and k is specified. ⋆ Unable to handle noise ⋆ Generally terminates at local optimum, not at global optimum. ⋆ It fails for categorical data. ⋆ It works just on numeric values.
Hierarchical Agglomerative Clustering	⋆ Can handle large dataset. ⋆ Reduced execution time due to parallelism ⋆ Increase in efficiency ⋆ It has embedded flexibility with regard to the level of granularity and it is easy to handle any forms of similarity or distance and applicable to any attribute types. ⋆ Performs clustering well with spherical data	⋆ Increase in efficiency is only linear. ⋆ It does not revisit once constructed clusters with the purpose of improvement.
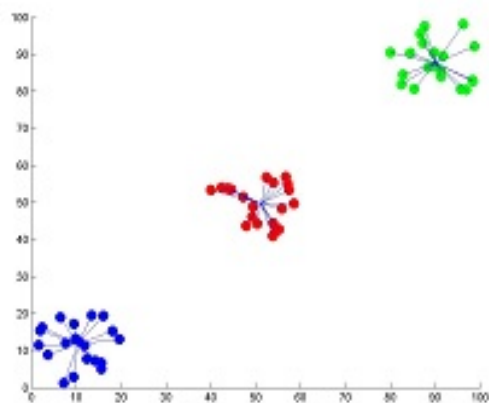
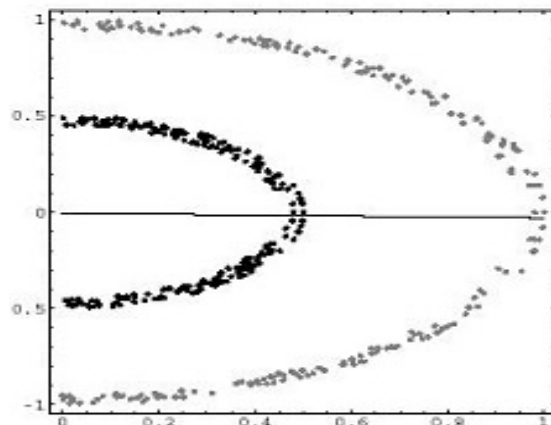Figure 1: Showing the result of k-means for 'N' = 60 and 'c'= 3

Figure 2: Showing the non-linear data set where k-means algorithm fails

4   Conclusion

In this paper analysis of classification and clustering algorithms has been made. The pros and cons of classification algorithms and clustering algorithms are listed. Under classification algorithms, Naive Bayes, SVM, KNN, Decision tree induction have been studied.

Decision Tree's algorithms are more accurate and they have less error rate and they are easier algorithms as compared to K-NN and Bayesian. C4.5 is the most prominent among decision tree algorithms.The steps of C4.5 are listed and its pros and cons are studied.

The challenge with clustering analysis is mainly that different clustering techniques give substantially different results on the same data. Moreover, there is no algorithm present which gives all the desired outputs. Under clustering algorithms $K-$ Means and Hierarchial clustering algorithms have been studied.

The time complexities of all the algorithms have been studied. The comparative study has shown that each algorithm has its own set of advantages and disadvantages as well as its own area of implementation. None of the algorithm can satisfy all constraints and criteria. Depending on application and requirements, specific algorithm can be chosen.

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Introduction to Stieltjes Transforms and its Applications

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Abstract: *Integral transform provides powerful operational method for solving initial value problem and initial boundary value problem for linear differential equations, difference equations, integral equations and many more arising problems in applied mathematics, mathematical physics and engineering science. In this project we touch on a type of integral transform called Stieltjes transform and Generalized Stieltjes transform, its definition, a few examples and its properties. We also look into Fourier-Stieltjes transforms and its applications to solve heat, wave and Laplace equations.*

Keywords: *Integral transforms, Stieltjes transforms, Generalized Stieltjes transforms, Fourier-Stieltjes transforms, Partial Differential Equations.*

AMS Subject Classification: 44A15.

1 Introduction

The concept of an integral transform follows directly from Fourier's theorem. Their widespread use in modern technology is due to the work of British mathematician, Oliver Heaviside. The integral transform of function $f(x)$ for $x \in [a, b]$ is defined as [1]

$$I\{f(x)\} = F(t) = \int_a^b K(x, t) f(x) \, dx$$

where $K(x, t)$ is called the Kernel of the transform. In integral transform, we transform the unknown function, $f(x)$, to a different function say $F(t)$.

In recent years integral transforms have become an essential tool in solving many mathematical problems. There are many other integral transformations like the Mellin transforms, Hankel transforms, Hilbert transforms, Stieltjes transforms, Laguerre transforms, Legendre transforms, Hermite transforms and many more, which find its applications in various branches of science. One form may be obtained from the other by transforming one of the coordinates or the functions. Depending on the type of the equation and the domain considered, a suitable integral transform can be chosen so as to simplify the computational techniques. Two of the most famous and widely used integral transforms are the Fourier transforms and Laplace transforms. The Laplace transform is used in solving initial value problems while the Fourier transform is used in solving boundary value problems, which are plenty in the analysis of time varying waveforms [2, 3, 4].

The Stieltjes transform is named after Thomas Joannes Stieltjes, who was a Dutch mathematician. He was a pioneer in the field of moment problems and contributed widely to the study of continued fractions and field analysis. The Riemann-Stieltjes integral which is named after Bernhard Riemann and Thomas Stieltjes, is the generalization of Riemann integral. The

Laplace Stieltjes transform is an integral transform applied only to the Fourier transform in it's utility in solving physical problems. It is useful in solving linear ordinary differential equations like those arising in the analysis of electronic circuits. The Stieltjes transform is applicable in music as stieltjies transform is used in Random matrix theory (RMI)[5, 6].

Definition of Stieltjes Transform, Generalized Stieltjes Transform and Fourier-Stiletjes Transform

The Stieltjes transform of a function $f(t)$ is derived by using the Laplace transform of a function $f(t)$ with respect to s. Thus Stieltjes transform of a function $f(t)$ is defined as

$$S\{f(t)\} = \tilde{f}(z) = \int\limits_0^\infty \frac{f(t)}{t+z} dt$$

The Generalized Stieltjes transform of a function $f(t)$ and a non-negative measure p on $[0, \infty)$ is defined as follows

$$S_g\{f(t)\} = \tilde{f}(z, p) = \int\limits_0^\infty \frac{f(t)}{(t+z)^p} dt$$

The measure p is considered to produce a convergent integral for each $z \in C$ and has a compact support to the Stieltjes functions, also known as Markov function.

The Fourier-Stieltjes transform is defined as [7]

$$FS\{f(t, x)\}(s, p) = \int\limits_0^\infty \int\limits_0^\infty e^{-ist}(x+y)^{-p} \, dt \, dx$$

where 't' and 'x' are called positive real numbers.

2 Properties of Stieltjes Transforms and Generalized Stieltjes Transforms

Theorem 1. *If* $S\{f(t)\} = \tilde{f}(z)$ *then* $S\{tf(t)\} = -z\tilde{f}(z) + \int\limits_0^\infty f(t)dt.$

Proof. Using the definition of Stieltjes transformation, we have

$$S\{tf(t)\} = \int\limits_0^\infty \frac{tf(t)}{t+z} dt$$

Adding and subtracting z in the numerator of the above integral, we have

$$S\{tf(t)\} = \int\limits_0^\infty f(t)dt - z\tilde{f}(z)$$

$\square$

Theorem 2. *If $S\{f(t)\} = \tilde{f}(z)$ then $S\left\{\dfrac{f(t)}{t+a}\right\} = \dfrac{1}{(a-z)}\left[\tilde{f}(z) - \tilde{f}(a)\right]$*

Proof. Using the definition of Stieltjes transformation, we have

$$S\left\{\frac{f(t)}{t+a}\right\} = \int\limits_0^\infty \frac{f(t)}{(t+a)(t+z)}dt$$

$$= \int\limits_0^\infty \frac{1}{(a-z)}\left[\frac{f(t)}{(t+z)}dt - \frac{f(t)}{(t+a)}dt\right]$$

$$= \frac{1}{a-z}[\tilde{f}(z) - \tilde{f}(a)]$$

$\square$

Theorem 3. *If $S\{f(t)\} = \tilde{f}(z)$ then $S\left\{\dfrac{1}{t}f\left(\dfrac{a}{t}\right)\right\} = \dfrac{1}{z}\tilde{f}\left(\dfrac{a}{z}\right)$, $a > 0$*

Proof. Using the definition of Stieltjes transformation, we have

$$S\left\{\frac{1}{t}f\left(\frac{a}{t}\right)\right\} = \int\limits_0^\infty \frac{f\left(\frac{a}{t}\right)}{t(t+z)}dt$$

$$= \int\limits_0^\infty \frac{1}{t(t+z)}f(\tau)\left(-\frac{a}{\tau^2}\right)d\tau \quad \left(\text{Substituting } \frac{a}{t} = \tau\right)$$

$$= \int\limits_0^\infty \frac{f(\tau)}{(a+z\tau)}d\tau$$

$$= \frac{1}{z}\tilde{f}\left(\frac{a}{z}\right)$$

$\square$

Theorem 4. *If $S_g\{f(t)\} = \tilde{f}(z, p)$ then $S_g\{tf(t)\} = \tilde{f}(z, p-1) - z\tilde{f}(z, p)$.*

Proof. Using the definition of generalized Stieltjes transformation, we have

$$S_g\{tf(t)\} = \int\limits_0^\infty \frac{tf(t)}{(t+z)^p}dt$$

Adding and subtracting z in the numerator, we have

$$S_g\{tf(t)\} = \int\limits_0^\infty \frac{f(t)}{(t+z)^{p-1}}dt - z\int\limits_0^\infty \frac{f(t)}{(t+z)^p}dt$$

$$= \tilde{f}(z, p-1) - z\tilde{f}(z, p)$$

$\square$

Theorem 5. *If $S_g\{f(t)\} = \tilde{f}(z,p)$ then*

$$S_g\{f'(t)\} = p\tilde{f}(z, p+1) - z^{-p}f(0) \quad for\ f(t) \to 0\ when\ t \to \infty.$$

Proof. Using the definition of Generalized Stieltjes transformation, we have

$$S_g\{f'(t)\} = \int_0^\infty f'(t)(t+z)^{-p}dt$$

$$= \left[\frac{f(t)}{(t+z)^p}\right]_0^\infty + p\int_0^\infty \frac{f(t)}{(t+z)^{p+1}}dt$$

$$= p\tilde{f}(z, p+1) - \frac{f(0)}{z^p} \qquad \text{(since } f(t) \to 0 \text{ as } t \to \infty)$$

$\square$

Theorem 6. *If $S_g\{f(t)\} = \tilde{f}(z,p)$ then* $S_g\left\{\int_0^t f(x)dx\right\} = \dfrac{\tilde{f}(z, p-1)}{(p-1)},\ Re(p) > 1.$

Proof. Let $g(t) = \displaystyle\int_0^t f(x)dx$

$g'(t) = f(t)$ and $g(0) = 0$

Then $S_g\{f(t), p\} = S_g\{g'(t), p\}$

$$= p\,\tilde{g}(z, p+1)$$

$$= p\,S_g\{g(t), p+1\}$$

$$S_g\{f(t), p-1\} = (p-1)S_g\{g(t), p\}$$

$$S_g\left\{\int_0^t f(x)dx\right\} = \frac{1}{p-1}\tilde{f}(z, p-1)$$

$\square$

3   Examples on Stieltjes Transforms and Generalized Stieltjes Transforms

Example 1: Find the Stieltjes transform of the following functions: [8]

(i) $f(t) = t^{\alpha-1}$ (ii) $f(t) = \dfrac{t^{\alpha-1}}{t+a}$ (iii) $f(t) = \dfrac{t}{t^2 + a^2}$

By the definition of Stieltjes transformation we have

(i)$S\{t^{\alpha-1}\} = \dfrac{1}{z}\displaystyle\int_0^\infty t^{\alpha-1}\left(1 + \frac{t}{z}\right)t^{-1}dt$

$$= z^{\alpha-1} \int_0^\infty \frac{x^{\alpha-1}}{(1+x)} dx \qquad \left(\text{Substituting } \frac{t}{z} = x \right)$$

$$= z^{\alpha-1} M\left\{ \frac{1}{1+x} \right\} \qquad \left(\text{since } M\left\{ \frac{1}{1+x} \right\} = \int_0^\infty \frac{x^{\alpha-1}}{1+x} dx \right)$$

$$= z^{\alpha-1} \pi \csc(\alpha\pi) \qquad \left(\text{since } M\left\{ \frac{1}{1+x} \right\} = \Gamma(p)\Gamma(1-p) \right)$$

(ii) $S\left\{ \dfrac{t^{\alpha-1}}{t+a} \right\} = \displaystyle\int_0^\infty \frac{t^{\alpha-1}}{t+z} dt$

$$= \frac{1}{(a-z)} \left[z^{-1} \int_0^\infty \left(1+\frac{t}{z}\right)^{-1} t^{\alpha-1} dt - a^{-1} \int_0^\infty \left(1+\frac{t}{a}\right)^{-1} t^{\alpha-1} dt \right]$$

Substituting $\dfrac{t}{z} = p$ and $\dfrac{t}{a} = q$ in the above integral, we obtain

$$S\left\{ \frac{t^{\alpha-1}}{t+a} \right\} = \frac{1}{(a-z)} \left[z^{\alpha-1} \int_0^\infty \frac{p^{\alpha-1}}{(1+p)} dp - a^{\alpha-1} \int_0^\infty \frac{q^{\alpha-1}}{(1+q)} dq \right]$$

$$= \frac{1}{(a-z)} \left[z^{\alpha-1} M\left\{ \frac{1}{1+p} \right\} dp - a^{\alpha-1} M\left\{ \frac{1}{1+q} \right\} dq \right]$$

$$= \frac{1}{(a-z)} [z^{\alpha-1} - a^{\alpha-1}] \pi \csc(\pi\alpha)$$

(iii) $S\left\{ \dfrac{t}{(t^2+a^2)} \right\} = \displaystyle\int_0^\infty \frac{t}{(t^2+a^2)(t+z)} dt$

$$= \frac{1}{a^2+z^2} \int_0^\infty \frac{1}{t+z} dt - \int_0^\infty \frac{t}{a^2+t^2} dt + \int_0^\infty \frac{z}{a^2+t^2} dt$$

$$= \frac{1}{a^2+z^2} \left[\frac{\pi a}{2} + z \log\left(\frac{z}{a}\right) \right]$$

Example 2: Find the generalized Stieltjes transform of the following functions:[9]

    (i) $t^{\alpha-1}$ (ii) $\exp(-at)$

By the definition of generalized Stieltjes transformation, we have

(i) $S_g\{t^{\alpha-1}\} = \displaystyle\int_0^\infty \frac{t^{\alpha-1}}{(t+z)^p} dt$

$$= z^{\alpha-p} \int_0^\infty \frac{u^{\alpha-1}}{(1+u)^p} z \, du \qquad (\text{Substituting } t = zu \,)$$

On substituting $x = \dfrac{u}{1+u}$ or $u = \dfrac{x}{1-x}$ in the above integral, we get

$$S_g\{t^{\alpha-1}\} = z^{\alpha-1} \int_0^\infty x^{\alpha-1}(1-x)^{p-\alpha-1}dx$$

$$= z^{\alpha-1}\beta(\alpha, p-\alpha)$$

$$= \frac{\Gamma(\alpha)\Gamma(p-\alpha)}{\Gamma(p)}z^{\alpha-1}$$

(ii) $S_g\{\exp(-at)\} = \displaystyle\int_0^\infty \frac{\exp(-at)}{(t+z)^p}dt$

$$= \int_0^\infty \frac{\exp(a(u-z))}{u^p}du \qquad \text{(Substituting } t+z=u\text{)}$$

$$= \exp(az)\int_0^\infty e^{-au}u^{-p}du$$

$$= a^{p-1}\exp(az)\Gamma(1-p) \qquad \text{(Substituting } au=x\text{)}$$

4   Applications of Fourier-Stieltjes Transforms

Mathematicians have more techniques to chose from while solving problems related to physical equations, integral equations and differential equations which have many applications in all fields of sciences. Using Fourier-Stieltjes transform we can obtain the solution for Wave equation, Heat flow equation and Laplace equation. The following are a few results which is required in solving the problems in Fourier-Stieltjes transforms [10, 7]

$$FS\{f_x(t,x)\} = pFS\{f(t,x)\} - k \tag{1}$$

$$FS\{f_n(t,x)\} = p^n FS\{f(t,x)\} - p^{n-1}k \tag{2}$$

$$FS\{f_t(t,x)\} = isFS\{f(t,x)\} - k \tag{3}$$

$$FS\{f_n(t,x)\} = (is)^n FS\{f(t,x)\} - (is)^n FSf(t,x) - (is)^{n-1}k \tag{4}$$

Application of Fourier-Stieltjes Transforms to Partial Differential Equations

Example 3: Find the Fourier-Stieltjes transform of the wave equation with the boundary conditions given by $f(t,0) = 0$ and $f(t,a) = 0$.

Consider the wave equation

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}$$

Taking the Fourier-Stieltjies transform of the above equation, we get

$$FS\{f_{tt}(t,x)\} = c^2 FS\{f_{xx}(t,x)\}$$

Applying the result (2) and (4) to the above expression, we obtain

$$(is)^2 FS\{f(t,x)\} - (is)k = c^2 p^2 FS\{f(t,x)\} - pk$$

$$D_x^2 FS\{f(t,x)\} = \frac{(is)^2}{c^2} FS\{f(t,x)\} - \frac{is}{c^2}k$$

$$\left(D_x^2 - \frac{(is)^2}{c^2}\right) FS\{f(t,x)\} = \frac{-is}{c^2}k$$

This is an ordinary differential equation with respect to x. After finding the complementary function and particular integral, the solution obtained is

$$FS\{f(t,x)\} = c_1 \cos\frac{sx}{c} + c_2 \sin\frac{sx}{c} + \frac{k}{is} \tag{5}$$

Substituting the boundary conditions we obtain $c_1 = -\frac{k}{is}$ and $c_2 = \frac{k}{is}\left(\cot\frac{s}{c}a - \csc\frac{s}{c}a\right)$. Thus equation (5) becomes

$$FS\{f(t,x)\} = \frac{k}{is}\left(\left(\cot\frac{s}{c}a - \csc\frac{s}{c}a\right)\sin\frac{s}{c} - \cos\frac{s}{c}x + 1\right)$$

Example 4: Find the Fourier-Stieltjies transform of the Laplace equation with the boundary conditions given by $f(t,0) = 0$ and $f(t,a) = 0$.

Consider the Laplace equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial t^2} = 0$$

By the definition of Fourier-Stieltjies integral transformation, we have

$$FS\{f(t,x)\} = \int_0^\infty \int_0^\infty e^{-ist}(x+y)^{-p} dt\, dx$$

Applying the Fourier-Stieltjies transform to the above equation, we get

$$FS\{f_{xx}(t,x)\} + Fs\{f_{tt}(t,x)\} = 0$$

Applying the result(2) and (4) to the above expression, we obtain

$$D^2 FS\{f(t,x)\} - pk + (is)^2 FS\{f(t,x)\} - (is)k = 0$$
$$D_x^2 FS\{f(t,x)\} + (is)^2 FS\{f(t,x)\} - isk = 0$$
$$(D_x^2 + (is)^2)FS\{f(t,x)\} = isk$$

This is an ordinary differential equation with respect to x. After finding the complementary function and particular integral, the solution obtained is

$$FS\{f(t,x)\} = c_1 e^{sx} + c_2 e^{-sx} + \frac{k}{is} \quad \left(\text{since } k = \int_0^\infty y^{-p} f(t,0)e^{-ist} dt\right) \tag{6}$$

Substituting the boundary conditions we obtain $c_1 = c_2 - \dfrac{k}{is}$ and $c_2 = \dfrac{k(1 - e^{sa})}{is(e^{sa} - e^{-sa})}$. Thus equation (6) becomes

$$FS\{f(t,x)\} = \frac{k}{is}\left(\frac{1 - e^{-sa}}{e^{-sa} - e^{sa}}\right)e^{sx} + \frac{k}{is}\left(\frac{1 - e^{sa}}{e^{sa} - e^{-sa}}\right)e^{-sx} + \frac{k}{is}$$

Example 5: Find the Fourier-Stieltjies transform of the heat equation with the boundary conditions given by $f(t,0) = 0$ and $f(t,a) = 0$.

Consider the heat equation

$$\frac{\partial f}{\partial t} = c^2 \frac{\partial^2 f}{\partial x^2} \quad \text{where } c^2 = \frac{1}{ps}$$

By the definition of Fourier-Stieltjies integral transformation we have

$$FS\{f(t,x)\}(s,p) = \int\limits_0^\infty \int\limits_0^\infty f(t,x)e^{-ist}(x+y)^{-p}$$

Taking the Fourier-Stieltjies transform of the above equation, we get

$$FS\{f_t(t,x)\}(s,p) = c^2 FS\{f_{xx}(t,x)\}(s,p)$$

Applying the result (2) and (4) to the above expression, we obtain

$$isFS\{f(t,x)\}(s,p) - k = c^2 P^2 FS\{f(t,x)\}(s,p) - pk$$

$$\left(D_x^2 - \frac{is}{c^2}\right)FS\{f(t,x)\}(s,p) = -\frac{k}{c^2}$$

This is an ordinary differential equation with respect to x. On solving we obtain

$$FS\{f(t,x)\}(s,p) = c_1 \cos\frac{\sqrt{s}}{\sqrt{ic}}x + c_2 \sin\frac{\sqrt{s}}{\sqrt{ic}}x + \frac{k}{is} \tag{7}$$

Using the boundary conditions we have $c_1 = \dfrac{-k}{is}$ and $c_2 = \dfrac{k}{is} + \left(\cot\dfrac{\sqrt{s}}{\sqrt{ic}}a - \csc\dfrac{\sqrt{s}}{\sqrt{ic}}a\right)$. Thus (7) becomes

$$FS\{f(t,x)\}(s,p) = \frac{k}{is}\left[\left(\cot\frac{\sqrt{s}}{\sqrt{ic}}a - \csc\frac{\sqrt{s}}{\sqrt{ic}}a\right)\sin\frac{\sqrt{s}}{\sqrt{ic}}x - \cos\frac{\sqrt{s}}{\sqrt{ic}}x + 1\right]$$

5   Conclusion

In this paper we have studied about an integral transform called Stieltjes transform and Generalized Stiletjes transform, through various examples and basic properties defined on them. We also learnt about solving the Wave equation, the Heat equation and the Laplace equation with Fourier-Stiletjes transform.

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Study of Adomian Decomposition Method to Nonlinear Equations

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Abstract: *In this paper, we discuss the general description of Adomian decomposition method. We solve some examples on nonlinear ordinary differential equation, partial differential equations and singular boundary value problems using Adomian decomposition method and by substitution of nonlinear parts of equation with Adomian polynomials and use step by step method for finding the solution and it has a faster convergence to the exact solution.*

Keywords: *Decomposition method, Adomian polynomials, Nonlinear equations, Quadratic equations, Singular boundary value problems.*

AMS Subject Classification: *41A30.*

1 Introduction

1.1 Adomian decomposition method

Adomian decomposition method is very useful for solving linear and nonlinear ordinary and partial differential equations and also algebraic equations. Many modifications were made on this method by numerous researchers in an attempt to improve the accuracy and extend the applications of this method [1].

The Adomian Decomposition method provides several significant advantages. It also handles the problem in a direct way without using linearization or any other restrictive assumptions. It provides an efficient numerical solution in the form of an infinite series that is obtained iteratively and converges to the exact solution using Adomian polynomials. The main advantage of this method is that it is capable of reducing the size of computational work without affecting the accuracy of the numerical solutions. The crucial aspect of this method is the Adomian polynomials which allow the solution for convergence of the nonlinear portion of the equation, without linearizing the system. This technique is very simple in an abstract formulation but the difficulty arises in calculating the polynomials and in proving the convergence of the series of functions [2].

These polynomials mathematically generalize to a Maclaurin series. We conclude that the Adomian decomposition method is an better approximation, which approximates the exact power series solution compared to other methods.

1.2 Adomian polynomials

Adomian polynomials decompose a function into a sum of components for a nonlinear operator. They are found to be uniformly convergent series of analytic functions and are derived

from unique formula. They provide the fastest rate of convergence of decomposition series for the nonlinear function [3]. The Adomian polynomial is given by,

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N(\sum_{i=0}^{n} \lambda^i x_i) \right]_{\lambda=0} \forall \, n = 0, 1, 2 \cdots .$$

2   Applications

1. Adomian decomposition method has extensive applications in fields such as physics, biology, chemistry and engineering.
2. Adomian method is applied to heat or mass transfer, particle transfer, nonlinear optics and the fermentation process.
3. Adomian method has been applied to several problems like to develop an analytic solution for a steady flow problem of a viscous incompressible fluid as governed by Navier-Stokes equations.
4. By comparing the ADM results of a simple problem of Poisson's equation with results of a numerical solution then the ADM to a reliable technique with less computational work.
5. Adomian method is applied for the linear and nonlinear Fokker-Planck equation.
6. Adomian method is used in solving some nonlinear oscillation equations and nonlinear heat transfer equations.
7. Adomian method is used in solving microchip laser problem equations.

3   Adomian decomposition method for quadratic equations

3.1   Method of solution

Consider a quadratic equation,
$$ax^2 + bx + c = 0. \tag{1}$$

The principle of the Adomian decomposition method of nonlinear equation is in the following form,

$$Lx + Nx = g$$
$$Lx = g - Nx$$

where, N is nonlinear operator and L is highest order derivative which is invertible hence L^{-1} is integral operator. Equation (1) can be rewritten as,

$$bx = -c - ax^2.$$

Dividing throughout by b we get,

$$x = -\left(\frac{c}{b}\right) - \left(\frac{a}{b}\right) ax^2.$$

The solution of x is now decomposed into components $x_0 + x_1 + \cdots$ where x_0 is taken as $\dfrac{-c}{b}$ and $x_1, x_2 \cdots$ are still to be identified.
Since $x_0 = -\dfrac{c}{b}$ we have,
$$x = x_0 - \frac{a}{b}x^2 \tag{2}$$

$$x = x_0 - \frac{a}{b} \sum_{n=0}^{\infty} A_n \tag{3}$$

The nonlinear term x^2 in equation(2) is replaced by $\sum_{n=0}^{\infty} A_n$, where, $A(x_0, x_1, \cdots x_n)$ refers to A_n where,

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N(\sum_{i=0}^{n} \lambda^i x_i) \right]_{\lambda=0} \quad \forall\, n = 0, 1, 2 \cdots \tag{4}$$

Equation (4) is known as Adomian polynomials.
The A_n's are given by,

$$A_0 = \frac{1}{0!} \frac{d^0}{d\lambda^0} \left[N \sum_{i=0}^{0} \lambda^i x_i \right]_{\lambda=0}$$

$$A_0 = \frac{d^0}{d\lambda^0} N(x_0) = x_0^2$$

$$A_1 = \frac{1}{1!} \frac{d}{d\lambda} \left[N \sum_{i=0}^{1} \lambda^i x_i \right]_{\lambda=0}$$

$$A_1 = x_1 N'(x_0) = 2x_0 x_1$$

$$A_2 = x_2 N'(x_0) + \frac{x_1^2}{2!} N''(x_0)$$

$$A_2 = x_1^2 + 2x_0 x_2$$

$$A_3 = x_3 N'(x_0) + \frac{2x_1 x_2}{2!} N''(x_0) + \frac{x_1^3}{3!} N'''(x_0)$$

$$A_3 = 2x_0 x_3 + 2x_1 x_2$$

$$A_4 = x_2^2 + 2x_0 x_4 + 2x_1 x_2$$

$$\vdots$$

$$\vdots$$

Substituting these $A_1, A_2, A_3, A_4, \cdots$ values in equation (3) then n-term approximation is given by,

$$\phi_n = \sum_{i=0}^{n-1} x_i \tag{5}$$

 Equation (5) gives the approximate value closer to the root [4].

4   Adomian decomposition method for singular boundary value problems

4.1   Method of solution

Consider the singular boundary value problems of the form [5]

$$(x^\alpha y')' = f(x, y) \tag{6}$$

subject to boundary conditions,

$$y(0) = A$$
$$y(1) = B$$

The linear differential operator L_{xx} is defined by

$$L_{xx}[\cdot] = \frac{d}{dx}\left(x^\alpha \frac{d}{dx}[\cdot]\right) \tag{7}$$

The inverse operator L_{xx}^{-1} is defined by

$$L_{xx}^{-1}[\cdot] = \int_0^x (x^{-\alpha}) \int_0^x [\cdot] dx dx \tag{8}$$

Operating with L_{xx}^{-1} on (7) we get ,

$$y = y(0) + \int_0^x (x^{-\alpha}) \int_0^x f(x, y) dx dx \tag{9}$$

Assuming that $f(x, y) = r(x)g(y)$ where $g(y)$ is the nonlinear term then

$$y = \sum_{n=0}^\infty y_n$$
$$g(y) = \sum_{n=0}^\infty A_n \tag{10}$$

where A_n is given by

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} g\left(\sum_{i=0}^\infty \lambda^i y_i\right)\right]_{\lambda=0} \tag{11}$$

Then,

$$y_0 = y(0)$$
$$y_{n+1} = \int_0^x x^{-\alpha} \int_0^x r(x) A_n dx dx$$

5   Examples and Discussions

Example 1: Find the root of the polynomial $(x - \frac{1}{4})(x - 100) = 0$.

Solution: Given, $x^2 - \frac{401}{4}x + 25 = 0$
The given quadratic equation is simplified in the form,

$$x = 0.2493 + 0.0099x^2$$

Let x^2 is replaced by Adomian polynomials then,

$$x = 0.2493 + 0.0099 \sum_{n=0}^\infty A_n = 0.2493 + 0.0099(A_0 + A_1 + A_2 + A_3 + A_4 + \cdots)$$

where,

$$A_0 = x_0^2$$
$$A_1 = 2x_0x_1$$
$$A_2 = x_1^2 + 2x_0x_2$$
$$\vdots$$

Then,

$$x_0 = 0.2493$$
$$A_0 = x_0^2 = 0.06215$$
$$x_1 = 0.0099A_0 = 0.0006$$
$$A_1 = 2x_0x_1 = 0.00030$$
$$x_2 = 0.0099A_1 = 0.0000003$$

The n-term approximation is given by

$$\phi_n = \sum_{i=0}^{n-1} x_i$$

$$\phi_1 = x_0 = 0.2493$$

$$\phi_2 = x_0 + x_1 = 0.2499$$

$$\phi_3 = x_0 + x_1 + x_2 = 0.2499$$

which is approximately close to the root $x = 0.25$

Example 2: Find the root of the polynomial $(x - \frac{1}{4})(x - 1) = 0$

Solution: Given, $x^2 - 1.25x + 0.25 = 0$
The given quadratic equation is simplified in the form :

$$x = 0.2 + 0.8x^2$$

Let x^2 is replaced by Adomian polynomial then,

$$x = 0.2 + 0.8\sum_{n=0}^{\infty} A_n = 0.2 + 0.8(A_0 + A_1 + A_2 + A_3 + A_4 + \cdots)$$

where,

$$A_0 = x_0^2$$
$$A_1 = 2x_0x_1$$
$$A_2 = x_1^2 + 2x_0x_2$$
$$A_3 = 2x_0x_3 + 2x_1x_2$$
$$A_4 = 2x_0x_4 + 2x_1x_3 + x_2^2$$
$$\vdots$$

Then,

$$x_0 = 0.2$$
$$A_0 = x_0^2 = 0.04$$
$$x_1 = 0.8A_0 = 0.032$$
$$A_1 = 2x_0x_1 = 0.0128$$
$$x_2 = 0.8A_1 = 0.01024$$
$$A_2 = x_1^2 + 2x_0x_2 = 0.00512$$
$$A_3 = 2x_0x_3 + 2x_1x_2 = 0.0022536$$
$$x_4 = 0.8A_3 = 0.001$$

The n-term approximation is given by

$$\phi_n = \sum_{i=0}^{n-1} x_i$$

$$\phi_1 = x_0 = 0.2$$

$$\phi_2 = x_0 + x_1 = 0.232$$

$$\phi_3 = x_0 + x_1 + x_2 = 0.242$$

$$\phi_4 = x_0 + x_1 + x_2 + x_3 = 0.246$$

$$\phi_5 = x_0 + x_1 + x_2 + x_3 + x_4 = 0.247$$

which is approximately close to the root $x = 0.25$

Example 3: Consider the nonlinear singular equation $y'' + \frac{1}{x}y' = y^3 - 3y^5$ subject to boundary conditions, $y(0) = 1, y(1) = \frac{1}{\sqrt{2}}$

Solution: Given, $y'' + \frac{1}{x}y' = y^3 - 3y^5, y_0 = 1$
The given equation is rewritten in the form,

$$(xy')' = x(y^3 - 3y^5)$$

The recurrence relation is given by,

$$y_{n+1} = \int_o^x x^{-1} \int_0^x [xA_n]dxdx$$

where the Adomian polynomial is given by,

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}N\left(\sum_{i=0}^n \lambda^i y_i\right)\right]$$

For $n = 0$,

$$A_0 = N(y_0) = y_0^3 - 3y_0^5 = (-2)$$
$$y_1 = \int_0^x x^{-1} \int_0^x [xA_0]dxdx$$

$$y_1 = -\frac{x^2}{2}$$

For $n = 1$,

$$A_1 = y_1 N'(y_0) = y_1[3y_0^2 - 15y_0^4] = 6x^2$$

$$y_2 = \int_0^x x^{-1} \int_0^x [xA_1]dxdx$$

$$y_2 = \frac{3x^4}{8}$$

For $n = 2$,

$$A_2 = y_2 N'(y_0) + \frac{y_1^2}{2}N''(y_0)$$

$$A_2 = y_2(3y_0^2 - 15y_0^4) + \frac{y_1^2}{2}(6y_0 - 60y_0) = -\frac{45}{4}x^4$$

$$y_3 = \int_0^x x^{-1} \int_0^x [xA_2]dxdx$$

$$y_3 = -\frac{5}{16}x^6$$

For $n = 3$,

$$A_3 = y_3 N'(y_0) + \frac{1}{2!}2y_1y_2 N''(y_0) + \frac{y_1^3}{3!}N'''(y_0)$$

$$A_3 = y_3(3y_0^2 - 15y_0^4) + y_1y_2(6y_0 - 60y_0^3) + \frac{y_1^3}{6}(6 - 180y_0^2) = \frac{280}{16}x^6$$

$$y_4 = \int_0^x x^{-1} \int_0^x [xA_3]dxdx$$

$$y_4 = \frac{35}{128}x^8$$

Then,

$$y = y_0 + \sum_{n=1}^{\infty} y_{n+1}$$

$$y = 1 + \left[\frac{-x^2}{2} + \frac{3x^4}{8} - \frac{5x^6}{16} + \frac{35}{128}x^8 - \cdots\right]$$

$$y = 1 + \sqrt{x}$$

Example 4: Consider the nonlinear singular equation $y'' + \frac{1}{2x}y' = e^y(\frac{1}{2} - e^y)$ subject to boundary conditions, $y(0) = \ln 2, y(1) = 0$

Solution: Given, $y'' + \frac{1}{2x}y' = e^y(\frac{1}{2} - e^y), y_0 = \ln 2$
The given equation is rewritten in the form,

$$(x^{\frac{1}{2}}y')' = x^{\frac{1}{2}}e^y(\frac{1}{2} - e^y)$$

The recurrence relation is given by,

$$y_{n+1} = \int_o^x x^{-\frac{1}{2}} \int_0^x [x^{\frac{1}{2}} A_n] dx dx$$

where the Adomian polynomial is given by ,

$$A_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} N \left(\sum_{i=0}^n \lambda^i y_i \right) \right]$$

For $n = 0$,

$$A_0 = N(y_0) = e^{y_0}(\frac{1}{2} - e^{y_0}) = (-3)$$

$$y_1 = \int_0^x x^{-\frac{1}{2}} \int_0^x [x^{\frac{1}{2}} A_0] dx dx$$

$$y_1 = \int_0^x x^{-\frac{1}{2}} \int_0^x x^{\frac{1}{2}}(-3) dx dx = -x^2$$

For $n = 1$,

$$A_1 = y_1 N'(y_0) = y_1 \left[e^{y_0}(-e^{y_0}) + (\frac{1}{2} - e^{y_0})e^{y_0} \right] = 7x^2$$

$$y_2 = \int_0^x x^{-\frac{1}{2}} \int_0^x [x^{\frac{1}{2}} A_1] dx dx = \int_0^x x^{-\frac{1}{2}} \int_0^x x^{\frac{1}{2}}(7x^2) dx dx$$

$$y_2 = \frac{x^4}{2}$$

For $n = 2$,

$$A_2 = y_2 N'(y_0) + \frac{y_1^2}{2} N''(y_0)$$

$$A_2 = y_2 \left[\frac{1}{2} e^{y_0}(1 - 4e^{y_0}) \right] + \frac{1}{2} y_1^2 \left[\frac{1}{2} e^{y_0}(1 - 8e^{y_0}) \right] = -11x^4$$

$$y_3 = \int_0^x x^{-\frac{1}{2}} \int_0^x [x^{\frac{1}{2}} A_2] dx dx = \int_0^x x^{-\frac{1}{2}} \int_0^x x^{\frac{1}{2}}(-11x^4) dx dx$$

$$y_3 = -\frac{x^6}{3}$$

For $n = 3$,

$$A_3 = y_3 N'(y_0) + \frac{1}{2!} 2 y_1 y_2 N''(y_0) + \frac{y_1^3}{3!} N'''(y_0)$$

$$A_3 = y_3 \left[\frac{1}{2} e^{y_0}(1 - 4e^{y_0}) \right] + y_1 y_2 \left[\frac{1}{2} e^{y_0}(1 - 8e^{y_0}) \right] + \frac{y_1^3}{6} \left[\frac{1}{2} e^{y_0}(1 - 16e^{y_0}) \right] = 15x^6$$

$$y_4 = \int_0^x x^{-\frac{1}{2}} \int_0^x [x^{\frac{1}{2}} A_3] dx dx = \int_0^x x^{-\frac{1}{2}} \int_0^x x^{\frac{1}{2}}(15x^6) dx dx$$

$$y_4 = \frac{x^8}{4}$$

Then,

$$y = y_0 + \sum_{n=1}^{\infty} y_{n+1}$$

$$y = \ln 2 + \left[-x^2 + \frac{x^4}{2} - \frac{x^6}{3} + \frac{x^8}{4} - \cdots \right]$$

$$y = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{n}$$

$$y = \ln 2 - \ln(1 + x^2)$$

$$y = \ln \left(\frac{2}{1 + x^2} \right)$$

6   Conclusion

In this paper the Adomian decomposition method yields the smaller root. The main advantage of this method is that it can be applied directly for all types of equations. The convergence is accurate locally and mainly in a neighborhood of the boundary points. An efficient approach is proposed to solve efficiently and easily a class of nonlinear singular two-point boundary value problems. The solving of nonlinear partial differential equations is to determine the exact solutions. It is simple powerful tool for obtaining the solutions without a large size of computations. It has a higher degree of accuracy.

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