ISSN xxxx-xxxx

Volume 3, Issue 3 September 2020

MES BULLETIN OF APPLIED SCIENCES

NAAC Accredited with 'A' Grade

EXECUTIVE EDITORS

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Volume 3, Issue 3 **September 2020**

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(Working Papers)

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TABLE OF CONTENTS

Alfven Inertial Internal Gravity Waves Propagating in an Exponentially Stratified Incompressible and Infinitely Conducting Fluid

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Absract: *Density stratification and gravity play important role in wave generation in non-homogeneous fluid. Gravity waves in homogeneous fluid exist only when there is a free surface, which is nothing but surface fluid discontinuity, i. e density stratification. Gravity acts as restoring force if there exist density stratification, which in turn leads to oscillations. For an incompressible fluid to be stable if the density of displaced fluid whose position is lower than old is greater and unstable if it is lesser. Thus oscillation or wave motion is possible only if the stratification is statically stable i.e density decreases with height. For stability of compressible fluid entropy decreases with elevation and wave motion exists only in stably stratified fluid. In this paper we have analyzed the effect of rotation and magnetic field on linear and nonlinear internal gravity waves called Alfven inertial internal gravity waves propagating in an exponentially stratified incompressible and infinitely conducting fluid. The problem is governed by nine nonlinear inhomogeneous PDE's which has been reduced to third order ODE's by using traveling wave solutions and some first integrals. The resulting system is analysed in Phase-plane. We have solved the same system by Rank matrix method. The Rank matrix gives us new solutions which are not obtained by Achala [2001].*

Keywords: *Travelling wave, Rotating stratified fluid, Inhomogeneous systems, Phase function, Internal Gravity waves, Rank matrix method.*

Subject Classification Code :

1 Introduction

Internal waves are waves which occur in the interior of a fluid where gravity is the restoring force. The density differences in the interior of the fluid are tiny compared to those at the surface which are present in the oceans and the atmosphere make it possible to have very large internal waves with large currents hence they can transport material in the ocean or atmosphere along with them. Inertial waves are a type of mechanical wave possible in rotating fluids commonly seen at the beach or in the bathtub. Inertial waves flow through the interior of the fluid, not at the surface and restoring force for inertial waves is the Coriolis force Most commonly they are observed in atmospheres, oceans, lakes, and laboratory experiments. Rossby waves, geostrophic currents, and geostrophic winds are examples of inertial waves. Inertial waves are also likely to exist in the molten core of the rotating Earth [1]. The linear theory of inertial waves is known well [2, 3] while the influence of nonlinear effects of wave interactions are subject of many recent theoretical and experimental studies.

Waves in electrically conducting fluids occurring as a result of the stability imparted by magnetic fields is known as hydromagnetic or Alfven waves [4]. They are found in plasmas or fluids with high electrical conductivity, such as the solar corona and Earth's magnetosphere and core. It is therefore a matter of considerable geophysical and astrophysical importance to understand and be able to quantitatively model such waves and their generalizations that occur when both magnetic fields and rotation are present.

The basic mechanism underlying waves in electrically conducting fluids permeated by magnetic fields was elucidated by Alfven [5]. He described a scenario whereby magnetic tension and inertial effects give rise to oscillations and travelling waves, which are known as Alfven waves in his honour. Lehnert [6] deduced that rapid rotation of a hydromagnetic system would lead to the splitting of plane Alfven waves into two circularly polarized, transverse, waves. He realized these would have very different timescales if the frequency of Inertial waves was much larger than that of pure Alfvén waves in the system. Here, such waves will be collectively be referred to as Magnetic Coriolis (MC) waves. Chandrasekhar [7] studied the effects of buoyancy on rotating hydromagnetic systems, though he focused primarily on axisymmetric motions. Braginsky [9, 10] described the influence of density stratification and convection driving non-axisymmetric waves naming these Magnetic Archimedes Coriolis (MAC) waves.

A more focused technical reviews of the subject are given by Roberts and Soward [11], Acheson and Hide [12, 13], Eltayeb [14, 15], Fearn Roberts and Soward [14], Proctor [17], Zhang and Schubert [18], Soward and Dormy [19]. Moffatt's monograph [20] is best material for essential reading.

Traveling wave solutions of inhomogeneous systems as quasi-simple waves has been studied extensively by many authors (see Courant and Friedrichs [21], Schindler [22]). The concept of simple integral elements in this context was introduced by Grundland [23] with a view to studying the properties of solutions which depend on the nature of the inhomogeneity. Extensive work in this regard has been carried out by many authors both for linear and nonlinear theory. Linear work is carried out by Bretherton [24], Booker and Bretherton [25], Jones [26], Acheson [27, 28, 29, 30], Grimshaw [31], Rudraiah and Venkatachalappa [32, 33, 34] and others.

Nonlinear theory was carried out by Seshadri and Sachdev [35] for acoustic gravity waves in compressible isothermal atmosphere, Venkatachalappa, Rudraiah and Sachdev [36], Rudraiah, Sachdev and Venkatachalappa [37] for rotating compressible stratified fluid, Venkatachalappa, Achala and Sachdev [38] for incompressible, rotating, stratified flows as limiting case of [36, 37]. Rudraiah and Venkatachalappa (1979) have studied the internal Alfven – inertial gravity waves with basic flow in different from zero subject to infinitesimal perturbations and obtained solutions near critical levels. Later Venkatachalappa, Rudraiah and Sachdev[36] have studied the propagation of linear and nonlinear traveling waves in a compressible rotating atmosphere. Venkatachalappa, Achala and Sachdev [38] have investigated the propagation of linear and nonlinear traveling waves in an exponentially stratified incompressible rotating fluid. Venkatachalappa, Rudraiah and Sachdev [37] have studied the propagation of linear and nonlinear hydro magnetic waves in an exponentially stratified incompressible medium. In this chapter we analyze the effect of rotation and magnetic field on linear and nonlinear internal gravity waves called Alfven inertial internal gravity waves propagating in an exponentially stratified incompressible and infinitely conducting fluid. The waves under study are governed by a system of nine nonlinear inhomogeneous PDEs. We seek travelling wave solutions of this system.

2 Basic Equations

The model considered is in Cartesian co-ordinate system with $x-$ and $y-$axes in the horizontal plane and $z-$ axes along vertical direction. The study is on quasi simple internal gravity waves in an incompressible, infinitely conducting stratified fluid rotating with uniform angular velocity Ω about a vertical axis in the presence of an applied variable magnetic field $H_0(z)$ in the $x-$ direction. The equations governing unsteady system are:

$$\rho\left[\frac{D\vec{q}}{Dt}+2\vec{\Omega}\times\vec{q}\right]+\nabla P-\rho\vec{g}-\mu(\vec{H}.\nabla)\vec{H}=0 \tag{1}$$

$$\frac{D\rho}{Dt}=0 \tag{2}$$

$$\nabla.\vec{q}=0 \tag{3}$$

$$\frac{D\vec{H}}{Dt}-\left(\vec{H}.\nabla\right)\vec{q}=0 \tag{4}$$

$$\nabla.\vec{H}=0 \tag{5}$$

where $\frac{D}{Dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}, P=p+\frac{\mu H^2}{2}$ is the total pressure, p the hydrodynamic pressure, $\vec{q}$ with components (u,v,w) the fluid velocity, ρ the density, $\vec{g}$ the acceleration due to gravity, $\vec{\Omega}$ the angular velocity, $\vec{H}$ the magnetic field with components (H_x,H_y,H_z) in the x,y,z directions respectively and μ is the permeability. We assume that the undisturbed fluid has density $\rho_0\left(z\right)$ and an applied magnetic field $H_0\left(z\right)$ of the form,

$$\rho_0\left(z\right)=\rho_c\quad\exp\left(-z/\vec{H}\right) \tag{6}$$

$$H_0\left(z\right)=H_c\quad\exp\left(-z/2\vec{H}\right) \tag{7}$$

where $\vec{H}$ is the scale height, ρ_c and H_c are the reference density and magnetic field at $z=0$. From equation (1) we find hat the basic pressure $\rho_0\left(z\right)$ is given by

$$p_0\left(z\right)=p_c\quad\exp\left(-z/\vec{H}\right) \tag{8}$$

where $p_c=g\vec{H}\rho_c$ is the hydrodynamic balance. We non dimensionalise the equations (1) - (5) using $H,\left(g/\vec{H}\right)^{\frac{1}{2}},\left(g\vec{H}\right)^{\frac{1}{2}},p_c e^{\left(-z/\vec{H}\right)},\rho_c e^{\left(-z/2\vec{H}\right)}$ as the scales for length, time, velocity, pressure, density and magnetic field respectively. We thus have

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-2\Omega v+\frac{1}{\rho}\frac{\partial P}{\partial x}-\frac{A^2}{\rho}\left(H_x\frac{\partial H_x}{\partial x}+H_y\frac{\partial H_x}{\partial y}+H_z\frac{\partial H_x}{\partial z}-\frac{H_xH_z}{2}\right)=0 \tag{9}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+2\Omega u+\frac{1}{\rho}\frac{\partial P}{\partial y}-\frac{A^2}{\rho}\left(H_x\frac{\partial H_y}{\partial x}+H_y\frac{\partial H_y}{\partial y}+H_z\frac{\partial H_y}{\partial z}-\frac{H_yH_z}{2}\right)=0 \tag{10}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}+\frac{1}{\rho}\frac{\partial P}{\partial x}+\left(1-\frac{p}{\rho}\right)-\frac{A^2}{\rho}\left(H_x\frac{\partial H_z}{\partial x}+H_y\frac{\partial H_z}{\partial y}+H_z\frac{\partial H_z}{\partial z}-\frac{H_z^2}{2}\right)=0 \tag{11}$$

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}-\rho w=0 \tag{12}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{13}$$

$$\frac{\partial H_x}{\partial t}+\left(u\frac{\partial H_x}{\partial x}+v\frac{\partial H_x}{\partial y}+w\frac{\partial H_x}{\partial z}\right)=0 \tag{14}$$

$$\frac{\partial H_y}{\partial t}+\left(u\frac{\partial H_y}{\partial x}+v\frac{\partial H_y}{\partial y}+w\frac{\partial H_y}{\partial z}\right)=0 \tag{15}$$

$$\frac{\partial H_z}{\partial t}+\left(u\frac{\partial H_z}{\partial x}+v\frac{\partial H_z}{\partial y}+w\frac{\partial H_z}{\partial z}\right)=0 \tag{16}$$

$$\frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}+\frac{\partial H_z}{\partial z}-\frac{H_z}{2}=0 \tag{17}$$

where $A^2=\dfrac{\mu H_c^2}{\rho_c g H}$ physically this represents non-dimensional Alfven velocity. We seek travelling wave solutions of the (9) - (17) in the form $u=u(\phi), v=v(\phi), w=w(\phi), \rho=\rho(\phi), P=P(\phi), H_x=H_x(\phi), H_y=H_y(\phi), H_z=H_z(\phi)$

$$\phi=\frac{x}{\lambda_1}+\frac{y}{\lambda_2}+\frac{z}{\lambda_3}-t \tag{18}$$

where $\lambda_1,\lambda_2,\lambda_3$ are arbitrary constants, that can be considered as wave length in x,y,z directions, the initial conditions are

$$u=v=w=H_y=H_z=0, p=\rho=H_x=1 \tag{19}$$

Hence equations (9)-(17) now become

$$u_\phi+2\Omega v-\frac{P_\phi}{\rho\lambda_1}+\frac{A^2}{2}\left[B\left(H_x\right)_\phi-\frac{H_zH_x}{2}\right]=0 \tag{20}$$

$$v_\phi+2\Omega u-\frac{P_\phi}{\rho\lambda_2}+\frac{A^2}{2}\left[B\left(H_y\right)_\phi-\frac{H_zH_y}{2}\right]=0 \tag{21}$$

$$w_\phi-\frac{P_\phi}{\rho\lambda_3}+\frac{p}{\rho}-1+\frac{A^2}{2}\left[B\left(H_z\right)_\phi-\frac{H_z^2}{2}\right]=0 \tag{22}$$

$$\rho_{phi}-\rho w=0 \tag{23}$$

$$\left(H_x\right)_\phi+Bu_\phi+\frac{wH_x}{2}=0 \tag{24}$$

$$\left(H_y\right)_\phi+Bv_\phi+\frac{wH_y}{2}=0 \tag{25}$$

$$\left(H_z\right)_\phi+Bw_\phi+\frac{wH_z}{2}=0 \tag{26}$$

$$B_\phi-\frac{H_z}{2}=0 \tag{27}$$

where $B = \frac{H_x}{\lambda_1} + \frac{H_y}{\lambda_2} + \frac{H_z}{\lambda_3}$, By suitably combining equations (20)-(27), we reduce this eighth order system to third order system,

$$w_\phi = \frac{-\overline{n}\lambda_3 (k-1) - 2\Omega\xi}{n\lambda_3 (1 - A^2 Q)} \tag{28}$$

$$k_\phi = kw - \frac{(1-k)}{n\lambda_3} - \frac{2\Omega\xi}{n} \tag{29}$$

$$\xi_\phi = \frac{2\Omega w}{\lambda_3 (1 - A^2 Q)} \tag{30}$$

where

$$k = \frac{p}{\rho}, \overline{n} = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}, n = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2} \tag{31}$$

the only singular point of the system (28)-(30) in (ξ, w, k) space is,

$$\xi = 0, k = 1, w = 0. \tag{32}$$

The solution can analysed near these singular points by linearising the system.

3 Methodology: Rank Matrix Method

One useful application of calculating the rank of a matrix is the computation of the number of solutions of a system of linear equations. According to the Roche-Capelli theorem, the system is inconsistent if the rank of the augmented matrix is greater than the rank of the coefficient matrix. If, on the other hand, ranks of these two matrices are equal, the system must have at least one solution. The solution is unique if and only if the rank equals the number of variables. Otherwise the general solution has *k* free parameters where *k* is the difference between the number of variables and the rank.

Let us think of a $r \times c$ matrix as a set of r row vectors, each having c elements; or let it be a set of c column vectors, each having r elements. The rank of a matrix is defined as the maximum number of linearly independent *column* vectors in the matrix or the maximum number of linearly independent *row* vectors in the matrix. For a matrix of order $r \times c$ matrix,

1. If $r < c$, then the maximum rank of the matrix is r.
2. If $r > c$, then the maximum rank of the matrix is c.
3. The rank of a matrix would be zero only if the matrix had no elements. If a matrix had even one element, its minimum rank would be one.
4. To find the rank of a matrix, we simply transform the matrix to its row echelon form and count the number of non-zero rows.

The system (20) to (27) can be written in the following form:

$$A_{ij} \frac{dU_j}{d\phi} = B_i \tag{33}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{\rho\lambda_1} & 0 & \frac{A^2B}{2} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{\rho\lambda_2} & 0 & 0 & \frac{A^2B}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{\rho\lambda_3} & 0 & 0 & 0 & \frac{A^2B}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ B & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & B & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & B & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\lambda_1} & \frac{1}{\lambda_2} & \frac{1}{\lambda_3} \end{bmatrix} U = \begin{bmatrix} u \\ v \\ w \\ \rho \\ p \\ H_x \\ H_y \\ H_z \end{bmatrix} B = \begin{bmatrix} \frac{A^2}{4}H_xH_z - 2\Omega v \\ \frac{A^2}{4}H_yH_z + 2\Omega u \\ \frac{A^2}{4}H_z^2 - \left(\frac{p}{\rho} - 1\right) \\ \rho w \\ -\frac{wH_x}{2} \\ -\frac{wH_y}{2} \\ -\frac{wH_z}{2} \\ \frac{H_z}{2} \end{bmatrix} \tag{34}$$

This system is then studied both when the coefficient matrix of the left-hand side of the algebraic system is of maximum rank and when it is lowers than that. In the latter case Kronecker-Capelli theorem leads to certain conditions on the unknown functions, which are, in fact, the intermediate integrals. The system does not have any solutions if the rank of A is less than that of $[A, B]$. Reducing the augmented matrix $[A, B]$ to echelon form we obtain,

$$\left[\begin{array}{cccccccc|c} 1 & 0 & 0 & -\frac{1}{\rho\lambda_1} & 0 & \frac{A^2B}{2} & 0 & 0 & \left\{\frac{A^2}{4}H_zH_x - 2\Omega v\right. \\ 0 & 1 & 0 & -\frac{1}{\rho\lambda_2} & 0 & 0 & \frac{A^2B}{2} & 0 & \left\{\frac{A^2}{4}H_zH_y - 2\Omega u\right. \\ 0 & 0 & 1 & -\frac{1}{\rho\lambda_3} & 0 & 0 & 0 & \frac{A^2B}{2} & \left\{\frac{A^2}{4}H_z^2 + \left(1 - \frac{p}{\rho}\right)\right. \\ 0 & 0 & 0 & -\frac{B}{\rho\lambda_1} & 1 & \frac{A^2B^2}{2} - 1 & 0 & 0 & \left\{\frac{A^2}{4}BH_z^2 + B\left(1 - \frac{p}{\rho}\right) + \frac{wH_z}{2}\right. \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \left\{\rho w\right. \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\lambda_2}\left(\frac{A^2B^2}{2} - 1\right) & \frac{1}{\lambda_3}\left(\frac{A^2B^2}{2} - 1\right) & 0 & \begin{cases} \frac{A^2B}{4\lambda_1}H_z^2 + \frac{B}{\lambda_1}\left(1 - \frac{p}{\rho}\right) + \frac{wH_z}{2\lambda_1} \\ -\frac{A^2B}{4\lambda_3}H_zH_x + \frac{2\Omega Bv}{\lambda_3} + \frac{wH_x}{2\lambda_3} \end{cases} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\lambda_1\lambda_3}\left(\frac{A^2B^2}{2} - 1\right) & -\frac{1}{\lambda_1\lambda_2}\left(\frac{A^2B^2}{2} - 1\right) & \begin{cases} \frac{A^2B}{4\lambda_2}H_z^2 + \frac{B}{\lambda_2}\left(1 - \frac{p}{\rho}\right) + \frac{wH_z}{2\lambda_2} \\ -\frac{A^2B}{4\lambda_3}H_zH_y - \frac{2\Omega Bu}{\lambda_3} - \frac{wH_y}{2\lambda_3} \end{cases} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & n\left(\frac{A^2B^2}{2} - 1\right) & \begin{cases} \overline{n}m + \overline{n}l + \frac{wH_z}{2}\overline{n} - \frac{A^2B}{4\lambda_2\lambda_3}H_zH_y \\ -\frac{2\Omega Bu}{\lambda_2\lambda_3} - \frac{wH_y}{2\lambda_2\lambda_3} + \frac{H_z}{2\lambda_3}\left(\frac{A^2B^2}{2} - 1\right) \\ -\frac{A^2B}{4\lambda_1\lambda_3}H_zH_x + \frac{2\Omega Bv}{\lambda_1\lambda_3} + \frac{wH_x}{2\lambda_1\lambda_3} \end{cases} \end{array}\right] \tag{35}$$

where,

$$\overline{n} = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} \tag{36}$$

$$n = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2} \tag{37}$$

$$m = \frac{A^2B}{4}H_z^2 \tag{38}$$

$$l = B\left(1 - \frac{p}{\rho}\right) \tag{39}$$

We can observe that the conditions for matrix A and $[A, B]$ to be of maximum rank equal to 8 are

$$\frac{A^2B^2}{2} - 1 \neq 0 \text{ or } n \neq 0. \tag{40}$$

Discussion of other conditions for full system is not so easy so now, let us discuss for the case $\lambda_3 \to \infty$

For rank to be 7 the conditions are:

$$\frac{A^2B^2}{2} - 1 = 0 \text{ and } m + l + \frac{wH_z}{2} = 0 \text{ or } \overline{n} = 0 \tag{41}$$

or

$$\frac{A^2B^2}{2} - 1 = 0 \text{ and } (H_z)_\phi = 0 \tag{42}$$

or

$$\frac{A^2B^2}{2} - 1 = 0 \text{ and } (H_z)_\phi = 0, m + l + \frac{wH_z}{2} = 0 \text{ or } \overline{n} = 0 \tag{43}$$

Case (i): $\frac{A^2B^2}{2} - 1 \neq 0$

Using this condition and solving (28) for the derivatives $U_{j,\phi}$, we get the solution which exactly matches with that obtained by Achala (Thesis 2001).
For rank to be seven, we have the following cases,

Case (ii): $\frac{A^2B^2}{2} - 1 = 0$ and $\overline{n}t + \overline{n}t + \frac{wH_z}{2}\overline{n} = 0$

Solving the system (20) to (27) using the above condition we get

$$B = \frac{\sqrt{2}}{A}; w = \frac{A}{\sqrt{2}}H_z. \tag{44}$$

These are very important new solutions in real plane.

Case (iii): $\frac{A^2B^2}{2} - 1 = 0$ and $(H_z)_\phi = 0$. Solving the system (4.20) to (4.27) using the above condition we get

$$\frac{H_x}{\lambda_1} + \frac{H_x}{\lambda_2} = C_1 \tag{45}$$

where C_1 is a constant.

Case (iv): $\frac{A^2B^2}{2} - 1 = 0$ and $(H_z)_\phi = 0, \overline{n}m + \overline{n}l + \frac{wH_z}{2}\overline{n} = 0$

Using the above condition, system (4.20) to (4.27) reduces to

$$w = \frac{A}{\sqrt{2}}H_z \tag{46}$$

The solutions obtained in **case (ii)** and **case(iv)** are same. These are new solutions and are not reported by any previous researchers. Thus we observe that the rank matrix gives all possible analytic solutions.

4 Inference

In this paper we obtain new analytic solutions to non-linear internal gravity waves called Alfven inertial internal gravity waves propagating in an exponentially stratified incompressible and infinitely conducting fluid by Rank matrix method. It is a very powerful method which gives all possible existing analytic solutions. We have obtained new conditional solutions of the problem which are given in equation (44) to (46).

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Educational Status of Women in Karnataka: An Inclusive Growth

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Absract: *This article analyses thepresent educational status of women in Karnataka with an approach towards inclusive growth. It is found that improvement in female literacy rates are higher than their male counterparts and the female literacy rates in rural areas and among SCs and STs needs to be enhanced.The GPI across literacy rates and GER have increased considerably indicating decline in gender disparities across the levels of education.The challenge before the state is to increase the GER to 30 per cent across higher education as specified by Planning Commission. The enhancement of female retention rates across 10th and 12th grade in the state is remarkable and females have scored comparatively higher to that of males across 10th and 12th grade, but there is decline in the passing percentages across 12th grade across the respective period.*

Keywords: *Gender Parity Index (GPI), Gross Enrolment Ratio (GER), Retention Rate and Inclusive Growth.*

1 Introduction

Education is one of the critical inputs which lays firm foundation for knowledge base of the society. It is necessary for enhancing and improving the human capital of any region. Theodore Schultz rightly remarks that investing in education will help in building up the human capital formation and formally organized education at the elementary, secondary and higher levels as one of the ways of developing human resources.Notable economist Lawrence Summers regards that investment in the education of girls may well be the highest return investment available in the developing world.

The Five-year plans especially 11th and 12th of Planning Commission of Government of India had encompassed the efforts to eliminate educational disparities for enhancing access, equity and quality at all stages of educational system in order to achieve faster inclusive growth. Similarly, World Bank Group has also dedicated its efforts for promoting girl's education and to improve gender equality. It is indicated very well in WBG's Gender Strategy 2016-2023: Gender Equality, Poverty Reduction and Inclusive Growth and Education Strategy 2020: Learning for All. According to 2011 Census, female population comprises of 49.3 per cent to the total in Karnataka state, but still they are behind their male counterparts in terms of education. Karnataka state has implemented noteworthy reforms in education sector in order to ensure access, equity and quality at all levels of education. Therefore, the objective of augmenting educational equalities with gender approach becomes crucial in the perspective of inclusive growth.

2 Literature Review and Objective

McDougall (2000) assessed the correlation between higher female literacy rate and lower gender gap in it but is still subject to deep regional variations. Biswal (2011) found out that regional, gender and social disparities in access and participation is a major concern even after visible progress in secondary education. Chanana (2011) critically analysed the policy approach towards exclusion and inclusion of women in higher education. Naik and Sharada (2013) found that districts of Karnataka state are marked with wide disparities in education even though few districts have recorded remarkable progress in educational development. Hong et.al (2019)examined the casual effect of reduction of inequality in gender education and enhancing female education and suggested that the effective way of promoting inclusive growth will be to expand women's educational opportunities. Extensive research is undertaken in the context of status of women's education in line with socio-economic development and inclusive growth. This article therefore tries to examine the present educational status of women in Karnataka with an approach towards inclusive growth.

3 Results and Discussion

This article henceforth attempts to track the educational progress of women in the Karnataka state by analyzing indicators namely literacy rates, share of enrolment and gross enrolment ratio at different levels of education, drop-out rate, retention rate and passing percentages across 10th and 12th grade.Literacy rate is one of the significant indicators for educational status. Table 1 presents the trends of gender-wise literacy rates in Karnataka state with regard to regions and social groups namely Scheduled Caste and Scheduled Tribes.

Table 1: Trends of Literacy rate in Karnataka (in per cent)

Particulars		2001	2011	Progress
Overall	Total	66.64	75.60	8.96
	Male	76.10	82.47	6.37
	Female	56.87	68.08	11.21
	GPI*	0.75	0.83	0.08
Rural	Total	59.33	68.86	9.53
	Male	70.45	77.61	7.16
	Female	48.01	59.71	11.70
	GPI	0.68	0.77	0.09
Urban	Total	80.58	86.21	5.63
	Male	86.66	90.04	3.38
	Female	74.12	81.36	7.24
	GPI	0.86	0.90	0.05
Scheduled Caste	Total	52.87	65.33	12.46
	Male	63.75	74.03	10.28
	Female	41.72	56.58	14.86
	GPI	0.65	0.76	0.11
Scheduled Tribe	Total	48.27	62.08	13.81
	Male	59.66	71.14	11.48
	Female	36.57	52.98	16.41
	GPI	0.61	0.74	0.13

*Note: *stands for Gender Parity Index*

Source: Extracted from Primary Census Abstract, Census Documents, GOI

It reveals that the overall literacy rate in the state has been enhanced by 8.96 points in 2011 over 2001. Gender-wise picture indicates that the male literacy rate is higher when compared to female literacy rate in both 2001 and 2011, but the progress over this period is seen highest across females. It increased by 11.21 points higher than the 6.37 points increase over male literacy rate. The gender parity index in literacy rates in the state has improved and stood at 0.83 during 2011.

Across rural and urban areas of Karnataka state also, it is revealed that the progress of female literacy rate is higher when compared to male literacy rates. It is important to note that the female literacy rates of rural areas in the state has not crossed 60 percent which becomes a challenge for fostering inclusive growth in education sector. The gender parity index in terms of literacy rates across rural areas also needs to be improved.

Further glance at Scheduled Caste and Scheduled Tribes reveals that the female literacy rates of these groups in the state are lagging behind when seen in comparison to their male counterparts. The progress of female literacy rates of Scheduled Tribes (16.41 points) is quite higher when compared to that of Scheduled Caste (14.86 points) during 2011 in the state. But the female literacy rates of these groups are still below 60 per cent. Therefore, there is urgency to enhance the female literacy rates of these groups for bettering the gender parity index of their literacy rates which is below 0.80.

Elementary education in Karnataka state covers the age group of 5 to 14 years, whereas secondary education comprises of 14 to 18 years and higher education includes above 18 years up to 24 years.During the period 2000-01 to 2019-20, the female enrolment across elementary education in the state progressed with an average annual growth rate of -0.4per cent whereas across secondary education growth rate of 1.4 per cent is noted. Noteworthy increase is noticed with regard to female enrolment across higher education registering average growth rate of around 10.3 per cent. The share of female enrolment to the total enrolment across different levels of education is quite promising when compared to its male counterparts (See Table 2). It reveals that the share of male enrolment to the total at elementary, secondary and higher education has recorded decline with 0.6 points, 5.4 points and 10.4 points respectively. On the contrary to it, the share of female enrolment to the total enrolment in the state have registered increase during the respective periods.The average share of female enrolment stood at 48.1 per cent across elementary education and 47.1 per cent across secondary education during the period 2000-01 to 2019-20. Whereas the average share of female enrolment across higher education was around 44.6 per cent during the period 2000-01 to 2018-19.

Table 2: Gender-wise share of enrolment in Karnataka (in per cent)

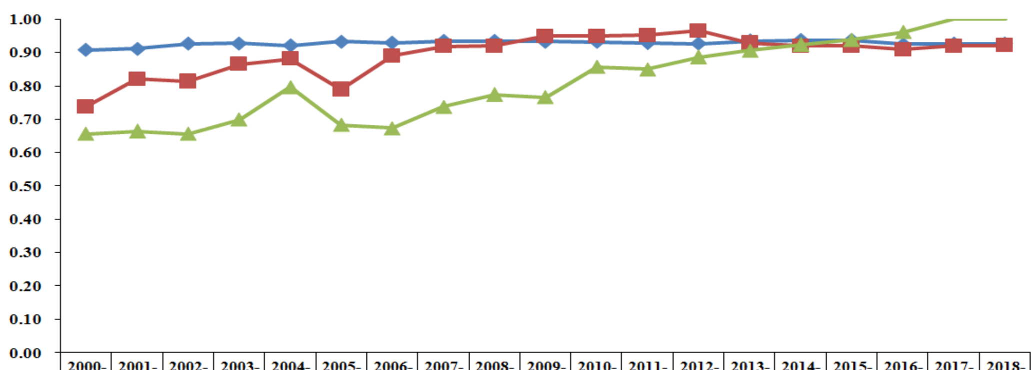
Years	Elementary Education (EE)		Secondary Education (SE)		Higher Education (HE)	
	Male	Female	Male	Female	Male	Female
2000-01	52.4	47.6	57.5	42.5	60.4	39.6
2001-02	52.3	47.7	54.9	45.1	60.1	39.9

2002-03	51.9	48.1	55.1	44.9	60.4	39.6
2003-04	51.9	48.1	53.6	46.4	58.8	41.2
2004-05	52.0	48.0	53.2	46.8	55.7	44.3
2005-06	51.7	48.3	55.9	44.1	59.4	40.6
2006-07	51.8	48.2	52.9	47.1	59.7	40.3
2007-08	51.7	48.3	52.1	47.9	57.5	42.5
2008-09	51.7	48.3	52.1	47.9	56.4	43.6
2009-10	51.7	48.3	51.3	48.7	56.6	43.4
2010-11	51.8	48.2	51.3	48.7	53.8	46.2
2011-12	51.8	48.2	51.2	48.8	54.1	45.9
2012-13	51.9	48.1	50.9	49.1	53.0	47.0
2013-14	51.7	48.3	51.8	48.2	52.5	47.5
2014-15	51.6	48.4	52.1	47.9	52.0	48.0
2015-16	51.6	48.4	52.1	47.9	51.6	48.4
2016-17	52.0	48.0	52.4	47.6	51.0	49.0
2017-18	51.9	48.1	52.1	47.9	50.0	50.0
2018-19	51.9	48.1	52.0	48.0	50.0	50.0
2019-20	51.8	48.2	52.1	47.9	-	-

Note: - Enrolment data not available

Source: Author's Calculations derived from Statistics of School Education, Statistics on Higher and Technical Education, Statistical Abstract of Karnataka and Economic Survey of Karnataka

Gender Parity Index (GPI) is calculated to understand the level of disparities between male and female across different educational indicators in the context of inclusive growth. GPI across enrolment at elementary, secondary and higher education is provided in Chart 1. It indicates that during the year 2000-01, the disparities across secondary and higher education were high, but less across elementary education. This is evident from GPI values which stood at 0.91 across elementary, 0.74 across secondary and 0.66 across higher education. Gradually over the period, the disparities have been reduced as we can notice increase in the GPI values. Further during 2018-19, the GPI value across elementary, secondary and higher education in the state have enhanced by 0.02 points, 0.18 points and 0.35 points respectively over the period 2000-01.



	2000-01	2001-02	2002-03	2003-04	2004-05	2005-06	2006-07	2007-08	2008-09	2009-10	2010-11	2011-12	2012-13	2013-14	2014-15	2015-16	2016-17	2017-18	2018-19
EE	0.91	0.91	0.93	0.93	0.92	0.93	0.93	0.94	0.93	0.93	0.93	0.93	0.93	0.93	0.94	0.94	0.92	0.93	0.93
SE	0.74	0.82	0.81	0.87	0.88	0.79	0.89	0.92	0.92	0.95	0.95	0.95	0.97	0.93	0.92	0.92	0.91	0.92	0.92
HE	0.66	0.66	0.66	0.70	0.80	0.68	0.67	0.74	0.77	0.77	0.86	0.85	0.89	0.91	0.92	0.94	0.96	1.00	1.00

Chart 1: Gender Parity Index (GPI) of education enrolment in Karnataka

Gross Enrolment Ratio (GER) is one of the suitable indicators to assess the extent of participation and access at different levels of education. It is the ratio of the number of persons enrolled (inclusive of over-aged and under-aged persons if any) to the total number of persons in the corresponding age-group. It is evident from Table 3 that from 2005-06 to 2018-19, there is high degree of participation and the increase in female GER across different levels of education is higher when compared to male GER. It is commendable in the light of inclusive growth in the state. With increase by 5.35 points across elementary education and 59.87-point increase across secondary education, the female GER in the state has crossed 100 per cent.

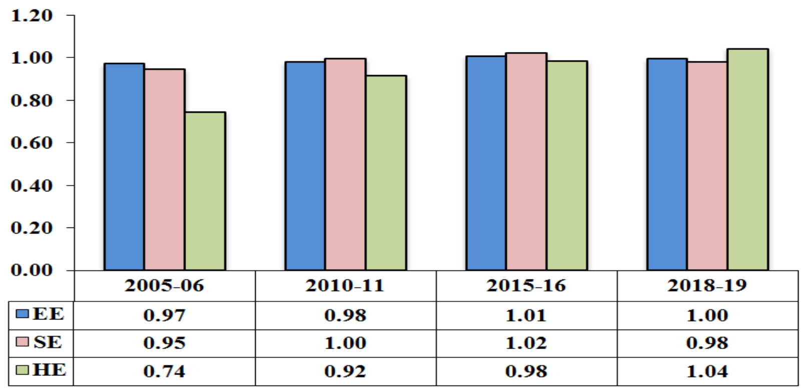
Table 3: Gross Enrolment Ratio (GER) in Karnataka

Years	Elementary Education (EE)		Secondary Education (SE)		Higher Education (HE)	
	Male	Female	Male	Female	Male	Female
2005-06	99.11	96.58	46.34	43.96	15.76	11.73
2010-11	100.20	98.30	57.90	57.80	26.55	24.34
2015-16	98.96	99.83	82.35	84.19	26.30	25.90
2018-19	102.32	101.93	105.68	103.83	28.20	29.40

Source: Reports of Karnataka School Education and MHRD reports titled "Statistics on School Education", "Statistics on Higher and Technical Education"

It seems that access across elementary and secondary education is not an issue, but the challenge before the state is to increase the GER across higher education. No doubt that, there in increase in the GER but neither the male GER nor the female GER across higher education have crossed 30 per cent in the state. This is the issue which needs to be addressed by the policy makers for fulfilling the Planning Commission target of 30 per cent GER to be achieved by 2020. The disparities in context of GER have seen reduction as it is evident from the increasing GPI values at all levels of education. It has increased by 0.03 points across elementary and secondary whereas by 0.30 points across higher education in the state (See Chart 2).

Chart 2: Gender Parity Index (GPI) of GER in Karnataka



Source: Author's calculations based on Table 3

Table 4 provides the contrary picture of gender-wise GER across the levels of education using the census data when compared to data published in the reports of Ministry of Human

Development Resource. As per 2011 Census figures, the GER across females pursuing elementary education stood at 76.18 per cent which is slightly lower to its male counterparts in the state. Still 23.82 per cent of females are left-out in this context. There is noteworthy increase in female GER across secondary education by 40.06 points during 2011 over 2001, but still 21.07 per cent are left-outs. The female GER pertaining to higher education has increased by 15.48 points in 2011 over 2001, but still it has not crossed 25 per cent. The major challenge before the state lies with regard to enhancement of GER across higher education in the state.

Table 4: Gross Enrolment Ratio of females in Karnataka based on Census data

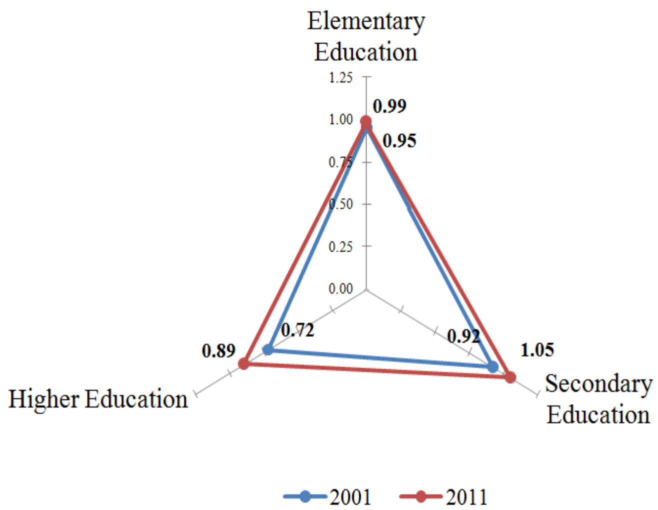
Years	Elementary Education (EE)		Secondary Education (SE)		Higher Education (HE)	
	Male	Female	Male	Female	Male	Female
2001	80.24	76.49	42.09	38.87	10.97	7.89
2011	77.27	76.18	74.90	78.93	26.12	23.37
Progress	-2.97	-0.31	32.81	40.06	15.15	15.48

Note: GER is calculated by dividing the enrolment at respective levels of education to that of population in the corresponding age groups

Source: Author's Calculations derived from Census documents

In the context of census data, the disparities in GER have considerably reduced across elementary and secondary education as the GPI value in 2011 stood at 0.99 and 1.05 respectively. Disparities in GER across higher education needs to be addressed by the policy makers as the GPI value of GER in 2011 stood at 0.89. (See Chart 3)

Chart 3: Gender Parity Index (GPI) of GER in Karnataka based on Census data



Source: Author's calculations based on Table 4

Drop-out rate is one of the serious challenges to be addressed in the context of inclusive growth and retention rate helps to determine the extent of inclusion in education. The Karnataka state has shown remarkable progress in terms of reducing drop-out rate and enhancing retention rate which are conversely related to each other.

Across elementary education, the state is successful in reducing its female drop-out rates by 40.8 points in 2018-19 over 2000-01, which is comparatively higher to that of 36.9 points reduction across males. The similar trend is witnessed with regard to reduction of drop-out rates across secondary education wherein 26.1-point reduction in female drop-out rates is noticed which is higher when compared to 23.9 points decline among male drop-out rates. (See Table 5) Therefore, the state can strive through its policies towards reaching 100 per cent retention rate and zero drop-out rates across both elementary and secondary education. This is quite crucial in fostering inclusive growth in education.

Table 5: Drop-out rates and Retention rates in Karnataka (in per cent)

Years	Drop-out rate		Retention rate	
	Male	Female	Male	Female
Elementary Education				
2000-01	49.0	53.5	51.0	46.5
2018-19	12.2	12.7	87.9	87.3
Secondary Education				
2000-01	33.7	34.0	66.3	66.0
2018-19	9.8	7.9	90.2	92.1

Source: Report on men and women in Karnataka 2010-11 and Karnataka State Education Report 2018-19

Passing percentages can be one of the indicators to enhance the transition rates and signifies access to upper levels of education. Gender-wise passing percentages across 10th grade and 12th grade is depicted in Table 6.

It reveals that the females have scored higher when compared to their male counterparts across both 10th and 12th grades in Karnataka. Even in terms of increase in the passing percentages across 10th grade over the years in the state, higher increase of 6.30 points is noticed among females as compared to 2.49 points increase among males during the period 2005-06 to 2020-21. On the contrary, the passing percentages across 12th grade have noticed decline over the years, but the decline of passing percentages among females is comparatively lesser to that of males. This poses question on the quality of education which is imparted and therefore there is an urgency to improve the passing percentages for enhancing the quality of education. Further it also ensures better enrolment across higher education.

Table 6: Passing Percentages in Karnataka (in per cent)

Years	10^{th} Grade		12^{th} Grade	
	Male	Female	Male	Female
2005-06	68.71	73.66	58.53	70.23
2010-11	76.40	80.50	48.80	64.20
2015-16	77.98	86.34	64.00	77.70
2019-20	69.02	80.06	55.29	68.24
2020-21	71.20	79.96	54.73	68.73
Progress	2.49	6.30	-3.80	-1.50

Source: Source: MHRD Report titled "Results of Secondary and Higher Secondary Examinations", KSEEB report titled "Analytical Statistics of SSLC results" and examresults.net

4 Conclusion

The findings of this study revealed that the Karnataka state has made gradual development in the context of eliminating gender disparities across elementary, secondary and higher education as evident from the Gender-Parity Index analysis. The cause of concerns in the light of inclusive growth are to enhance the female literacy rates of rural areas, SCs and STs, to boost the female Gross enrolment rate at higher education and to increase the passing percentages across 10th and 12th grade in the state.

The policy makers of the state should continue the specific enrolment drive campaigns at elementary level such as pre-matric and post-matric scholarships, Akshara Dasoha, Ksheera Bhagya, Nali-Kali, Keli-Kali, Chinnara Angala, distribution of free textbooks, uniforms, school bags etc so as to sustain the retention rates and to fulfill the goal of universalization of elementary education under Right to Education Act. Similarly other initiatives such as Capacity building programmes, Dhaklathi Andholana, fee exemption for girl students studying in Government PU colleges, free laptop scheme for 12th pass students, NAAC Accreditation, Career counseling under United Nations Development Programme named DISHA, etc will go long way in enhancing participation across secondary and higher education and to improve the quality of education in the state. It will support to upgrade their employability skills which further leads to securing of their livelihoods.

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A Study on Jacobi's Two Square Theorem

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Absract: *The survey of Jacobi's sum of squares and calculation of the number of representations of a given positive integer into sum of two squares based on divisor function is studied. Calculating manually, the number of representations, may although seem to be interesting at the beginning for smaller numbers, but later on becomes a tedious job for larger numbers. Fortunately we have formulae for sum of squares given by Jacobi, which directly gives us the number of representations. This paper consists the proof of theorem given by M. D. Hirschhorn [1] and another proof using Ramanujan's $_1\psi_1$ summation formula [2]. A MATLAB program for the calculation of the same is written.*

Keywords: *Jacobi's sum of squares, divisor function, Jacobi's triple product identity [3], Ramanujan's $_1\psi_1$ summation formula [4].*

Subject Classification Code :

1 Introduction

In 1640 Fermat stated that a prime p is the sum of two squares if and only if $p \equiv 1 \pmod 4$ and this was eventually proved by Euler in 1747. In 1801 Guass showed that the number n is the sum of two square if and only if the squarefree part of n has no divisor congruent to -1 modulo 4. In 1829 Jacobi proved that the number of representations of n as a sum of two square is 4 times the difference between the number of divisors of n congruent to 1 modulo 4 and the number of divisors of n congruent to 3 modulo 4 [1].

Definition:

For positive integers n and k, let $r_k(n)$ denote the representations of n as a sum of k squares, where representations with different orders and different signs are counted as distinct. By convention, $r_k(0) = 1$ [5].

Examples:

$r_2(2) = 4$, because $2 = 1^2 1^2 = 1^2 + (-1)^2 = (-1)^2 + (1)^2 = (-1)^2 + (-1)^2$;
$r_2(9) = 4$, because $9 = (3)^2 + (0)^2 = (0)^2 + (3)^2 = (-3)^2 + (0)^2 = (0)^2 + (-3)^2$;
$r_2(7) = 0$, because there are no ways we can write 7 as a sum of two squares.

Theorem 1.1. *For each positive integer n,*

$$r_2(n) = 4 \sum_{\substack{d|n \\ d \text{ odd}}} (-1)^{\frac{d-1}{2}}. \tag{1}$$

We can state this theorem in the alternative formulation as,

$$r_2(n) = 4(d_{1,4}(n) - d_{3,4}(n)) \tag{2}$$

where $d_{j,k}(n)$ denotes the number of positive divisors d of n such that $d \equiv j \pmod k$. Immediately deducible from (2) *is the well known theorem that every prime p congruent to 1 modulo 4 can be represented as sum of two squares [6].*

Proof. Let us consider the Jacobian triple product identity $\forall\ z \neq 0, |q| < 1$

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (-zq; q^2)_\infty(-q/z; q^2)_\infty(q^2; q^2)_\infty, \tag{3}$$

putting $z = -z^2 q$, $q^2 = q$ in equation (3)

$$(z^2 q; q)_\infty (z^{-2}; q)_\infty (q; q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n z^{2n} q^{\frac{n^2+n}{2}}, \tag{4}$$

where $(z^{-2}; q)_\infty = (1 - z^{-2})(z^{-2}q; q)_\infty$, then equation (4) can be rewritten as,

$$(z^2 q; q)_\infty (1 - z^{-2})(z^{-2} q; q)_\infty (q; q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n z^{2n} q^{\frac{n^2+n}{2}}, \tag{5}$$

By multiplying both sides of equation (5) by z we get,

$$(z - 1/z)(z^2 q; q)_\infty (z^{-2} q; q)_\infty (q; q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n z^{2n+1} q^{\frac{n^2+n}{2}}. \tag{6}$$

From R.H.S. of equation (6) we have,

$$\sum_{n=-\infty}^{\infty} (-1)^n z^{2n+1} q^{\frac{n^2+n}{2}} = \left(\sum_{\substack{n=-\infty \\ \text{n even}}}^{\infty} + \sum_{\substack{n=-\infty \\ \text{n odd}}}^{\infty} \right) (-1)^n z^{2n+1} q^{\frac{n^2+n}{2}}. \tag{7}$$

Replacing $n = 2n$ and $n = 2n - 1$ in R.H.S. of equation (7) we get,

$$\begin{aligned} \text{R.H.S.} &= \sum_{n=-\infty}^{\infty} z^{4n+1} q^{2n^2+n} - \sum_{n=-\infty}^{\infty} z^{4n-1} q^{2n^2-n} \\ &= z(-z^4 q^3; q^4)_\infty (-z^{-4} q; q^4)_\infty (q^4; q^4)_\infty \\ &- 1/z(-z^4 q; q^4)_\infty (-z^{-4} q^3; q^4)_\infty (q^4; q^4)_\infty. \end{aligned} \tag{8}$$

where we applied the equation (3) two additional times.

We next use logarithmic differentiation to differentiate both sides of the equation (6) with respect to z and set $z = 1$. (Note that on the far left side of (6) the differentiation of the infinite products is unnecessary, because when $z = 1$, the factor $z - 1/z = 0$).

Now consider the R.H.S. of equation (6) which is further simplified as equation (8),

$$\begin{aligned}\text{R.H.S.} =& z(-z^4q^3;q^4)_\infty(-z^{-4}q;q^4)_\infty(q^4;q^4)_\infty \\ &- 1/z(-z^4q;q^4)_\infty(-z^{-4}q^3;q^4)_\infty(q^4;q^4)_\infty \\ =& u - v \\ \frac{d}{dz}\big(\text{R.H.S.}\big) =& \frac{du}{dz} - \frac{dv}{dz} \end{aligned} \tag{9}$$

Let,

$$\begin{aligned} u &= z(-z^4q^3;q^4)_\infty(-z^{-4}q;q^4)_\infty(q^4;q^4)_\infty \\ \log u &= \log\Big[z(-z^4q^3;q^4)_\infty(-z^{-4}q;q^4)_\infty(q^4;q^4)_\infty\Big] \\ \frac{1}{u}\frac{du}{dz} &= \frac{1}{z} + \left[\frac{1}{1+z^4q^3}(4z^3q^3) + \frac{1}{1+z^4q^7}(4z^3q^7) + ...\right] \\ &+ \left[\frac{1}{1+z^{-4}q}(-4z^{-5}q) + \frac{1}{1+z^{-4}q^5}(-4z^{-5}q^5) + ...\right] \\ &= 1 + \sum_{n=1}^{\infty}\frac{4q^{4n-1}}{1+q^{4n-1}} - \sum_{n=1}^{\infty}\frac{4q^{4n-3}}{1+q^{4n-3}} \\ \frac{du}{dz} &= (-q^3;q^4)_\infty(-q;-q^4)_\infty(q^4;q^4)_\infty \times \left\{1 + \sum_{n=1}^{\infty}\frac{4q^{4n-1}}{1+q^{4n-1}} - \sum_{n=1}^{\infty}\frac{4q^{4n-3}}{1+q^{4n-3}}\right\} \end{aligned} \tag{10}$$

Now take the second term,

$$\begin{aligned} v &= \frac{1}{z}(-z^4q;q^4)_\infty(-z^{-4}q^3;q^4)_\infty(q^4;q^4)_\infty \\ \log v &= \log\left[\frac{1}{z}(-z^4q;q^4)_\infty(-z^{-4}q^3;q^4)_\infty(q^4;q^4)_\infty\right] \\ \frac{1}{v}\frac{dv}{dz} &= \left(z.\frac{-1}{z^2}\right) + \left[\frac{1}{1+z^4q}(4z^3q) + \frac{1}{1+z^4q^5}(4z^3q^5) + ...\right] \\ &+ \left[\frac{1}{1+z^{-4}q^3}(-4z^{-5}q^3) + \frac{1}{1+z^{-4}q^7}(-4z^{-5}q^7) + ...\right] \\ &= -1 + \sum_{n=1}^{\infty}\frac{4q^{4n-3}}{1+q^{4n-3}} - \sum_{n=1}^{\infty}\frac{4q^{4n-1}}{1+q^{4n-1}} \\ \frac{dv}{dz} &= (-q;q^4)_\infty(-q^3;-q^4)_\infty(q^4;q^4)_\infty \times \left\{-1 + \sum_{n=1}^{\infty}\frac{4q^{4n-3}}{1+q^{4n-3}} - \sum_{n=1}^{\infty}\frac{4q^{4n-1}}{1+q^{4n-1}}\right\} \end{aligned} \tag{11}$$

Substituting both equations (10) and (11) in equation (9) we obtain,

$$\begin{aligned} \frac{d}{dz}\big(\text{R.H.S.}\big) =& (-q^3;q^4)_\infty(-q;-q^4)_\infty(q^4;q^4)_\infty \times \left\{1 + \sum_{n=1}^{\infty}\frac{4q^{4n-1}}{1+q^{4n-1}} - \sum_{n=1}^{\infty}\frac{4q^{4n-3}}{1+q^{4n-3}}\right\} \\ &+ (-q;q^4)_\infty(-q^3;-q^4)_\infty(q^4;q^4)_\infty \times \left\{1 - \sum_{n=1}^{\infty}\frac{4q^{4n-3}}{1+q^{4n-3}} + \sum_{n=1}^{\infty}\frac{4q^{4n-1}}{1+q^{4n-1}}\right\} \end{aligned}$$

$$= (-q^3;q^4)_\infty(-q;-q^4)_\infty(q^4;q^4)_\infty \times \left\{2+2\sum_{n=1}^{\infty}\frac{4q^{4n-1}}{1+q^{4n-1}} - 2\sum_{n=1}^{\infty}\frac{4q^{4n-3}}{1+q^{4n-3}}\right\}$$

$$\frac{d}{dz}\big(\text{R.H.S.}\big) = 2(-q^3;q^4)_\infty(-q;-q^4)_\infty(q^4;q^4)_\infty \times \left\{1-4\sum_{n=1}^{\infty}\left(\frac{q^{4n-3}}{1+q^{4n-3}} - \frac{q^{4n-1}}{1+q^{4n-1}}\right)\right\} \tag{12}$$

Similarly taking logarithmic differentiation on L.H.S. of equation (6) we get,

$$\frac{d}{dz}\big(\text{L.H.S.}\big) = 2(q;q)_\infty^3 \tag{13}$$

Substituting equations (12) and (13) in equation (6) we get,

$$2(q;q)_\infty = 2(-q^3;q^4)_\infty(-q;-q^4)_\infty(q^4;q^4)_\infty \times \left\{1-4\sum_{n=1}^{\infty}\left(\frac{q^{4n-3}}{1+q^{4n-3}} - \frac{q^{4n-1}}{1+q^{4n-1}}\right)\right\} \tag{14}$$

Now divide equation (14) by 2 which results in,

$$\begin{aligned}(-q;q)_\infty^2(q;q)_\infty &= (-q;q)_\infty(q^2;q^2)_\infty\\ &= (-q;q^2)_\infty(-q^2;q^2)_\infty(q^2;q^2)_\infty\\ &= (-q;q^2)_\infty(q^4;q^4)_\infty\\ &= (-q^3;q^4)_\infty(-q;q^4)_\infty(q^4;q^4)_\infty,\end{aligned}$$

to deduce that,

$$\frac{(q;q)_\infty^2}{(-q;q)_\infty^2} = 1-4\sum_{n=1}^{\infty}\left(\frac{q^{4n-3}}{1+q^{4n-3}} - \frac{q^{4n-1}}{1+q^{4n-1}}\right) \tag{15}$$

By the special cases of Ramanujan's theta function we have,

$$\varphi^2(-q) = \frac{(q;q)_\infty^2}{(-q;q)_\infty^2} = (q;q^2)_\infty^4(q^2;q^2)_\infty^2. \tag{16}$$

Using equation (16) in equation (15) and replacing q by $-q$ we conclude that,

$$\begin{aligned}\varphi^2(q) &= 1+4\sum_{n=1}^{\infty}\left(\frac{q^{4n-3}}{1-q^{4n-3}} - \frac{q^{4n-1}}{1-q^{4n-1}}\right)\\ &= 1+4\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{d|n\\ d\equiv 1 \pmod 4}} 1 - \sum_{\substack{d|n\\ d\equiv 3 \pmod 4}} 1\Bigg)q^n\end{aligned} \tag{17}$$

By equating the coefficients of $q^n, n \geq 1$, on both sides of equation (17) we get,

$$r_2(n) = 4(d_{1,4}(n) - d_{3,4}(n))$$

which is an alternate formulation of equation (2). □

2 Sum of two squares theorem using Ramanujan's ${}_1\psi_1$ Summation formula

Theorem 2.1. *Using corollary of Ramanujan's ${}_1\psi_1$ summation formula i.e.,*

$$1+\sum_{n=1}^{\infty}\frac{(1/\alpha;q^2)_n(-\alpha q)^n}{(\beta q^2;q^2)_n}z^n+\sum_{n=1}^{\infty}\frac{(1/\beta;q^2)_n(-\beta q)^n}{(\alpha q^2;q^2)_n}z^n$$
$$=\frac{(q^2;q^2)_\infty(\alpha\beta q^2;q^2)_\infty(-qz;q^2)_\infty(-q/z;q^2)_\infty}{(\alpha q^2;q^2)_\infty(\beta q^2;q^2)_\infty(-\alpha qz;q^2)_\infty(-\beta q/z;q^2)_\infty} \tag{18}$$

Proof. Substitute $\alpha=\beta=-1$ and $z=1$,

$$1+\sum_{n=1}^{\infty}\frac{(-1;q^2)_n(q)^n}{(-q^2;q^2)_n}z^n+\sum_{n=1}^{\infty}\frac{(-1;q^2)_n(q)^n}{(-q^2;q^2)_n}z^n=\frac{(q^2;q^2)_\infty(q^2;q^2)_\infty(-q;q^2)_\infty(-q;q^2)_\infty}{(-q^2;q^2)_\infty(-q^2;q^2)_\infty(q;q^2)_\infty(q;q^2)_\infty}. \tag{19}$$

Now consider the R.H.S. of equation (19)

$$\frac{(q^2;q^2)_\infty(q^2;q^2)_\infty(-q;q^2)_\infty(-q;q^2)_\infty}{(-q^2;q^2)_\infty(-q^2;q^2)_\infty(q;q^2)_\infty(q;q^2)_\infty}=\frac{(q^2;q^2)_\infty^2(-q;q^2)_\infty^2}{(-q^2;q^2)_\infty^2(q;q^2)_\infty^2}$$
$$=(-q;q^2)_\infty^4(q^2;q^2)_\infty^2$$
$$=\varphi^2(q). \tag{20}$$

By the Ramanujan's θ function we have,

$$f(a,b)=\sum_{n=-\infty}^{\infty}a^{(n(n+1))/2}b^{(n(n-1))/2}$$
$$=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{21}$$

Putting $a=q$ and $b=q$ in equation (21), we obtain

$$f(q;q)=\sum_{n=-\infty}^{\infty}q^{n^2}$$
$$=(-q;q^2)_\infty(-q;q^2)_\infty(q^2;q^2)_\infty$$
$$=(-q;q^2)_\infty^2(q^2;q^2)_\infty$$
$$=\varphi(q),$$

Similarly we obtain

$$(-q;q^2)_\infty^4(q^2;q^2)_\infty^2=\varphi^2(q).$$

Now consider L.H.S. of equation (19)

$$1+\sum_{n=1}^{\infty}\frac{(-1;q^2)_n(q)^n}{(-q^2;-q^2)_n}+\sum_{n=1}^{\infty}\frac{(-1;q^2)_n(q)^n}{(-q^2;-q^2)_n}=1+2\sum_{n=1}^{\infty}\frac{(-1;q^2)_n(q)^n}{(-q^2;-q^2)_n}$$
$$=1+2\sum_{n=1}^{\infty}\frac{(1+1)(1+q^2)(1+q^4)....}{(1+q^2)(1+q^4)(1+q^6)....(1+q^{2n})}q^n$$

$$= 1 + 2 \times 2 \sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} = 1 + 4 \sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} \tag{22}$$

From equations (20) and (22) we have,

$$1 + 4 \sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} = (-q;q^2)_\infty^4 (q^2;q^2)_\infty^2 = \varphi^2(q) \tag{23}$$

On the other hand we have,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} &= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (-1)^m q^{n+2mn} \\ &= \sum_{m=0}^{\infty} (-1)^m \sum_{n=1}^{\infty} q^{(2m+1)n} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m (q^{2m+1})}{1-q^{2m+1}} \end{aligned} \tag{24}$$

Substituting equation (24) in equation (23) we get,

$$1 + 4 \left(\sum_{m=0}^{\infty} \frac{(-1)^m q^{2m+1}}{1-q^{2m+1}} \right) = \varphi^2(q). \tag{25}$$

We see that we have arrived at the first equality of (17). The remainder of the proof then follows as discussed in the previous theorem. □

3 Examples

Using equation (2), for $n = 1$ and $d = 1$ we get

$$\begin{aligned} r_2(1) = 4(1-0) = 4 \Longrightarrow &(1)^2 + (0)^2, \\ &(-1)^2 + (0)^2, \\ &(0)^2 + (1)^2, \\ &(0)^2 + (-1)^2. \end{aligned}$$

For $n = 2$, $d = 1, 2$

$$\begin{aligned} r_2(2) = 4(1-0) = 4 \Longrightarrow &(1)^2 + (1)^2, \\ &(-1)^2 + (1)^2, \\ &(1)^2 + (-1)^2, \\ &(-1)^2 + (-1)^2. \end{aligned}$$

For $n = 3$, $d = 1, 3$

$$r_2(3) = 4(1-1) = 0.$$

For $n = 4$, $d = 1, 2, 4$

$$\begin{aligned} r_2(4) = 4(1-0) = 4 \Longrightarrow & (2)^2 + (0)^2, \\ & (-2)^2 + (0)^2, \\ & (0)^2 + (2)^2, \\ & (0)^2 + (-2)^2. \end{aligned}$$

For $n = 5$, $d = 1, 5$

$$\begin{aligned} r_2(5) = 4(2-0) = 8 \Longrightarrow & (2)^2 + (1)^2, \\ & (-2)^2 + (1)^2, \\ & (2)^2 + (-1)^2, \\ & (-2)^2 + (-1)^2, \\ & (1)^2 + (2)^2, \\ & (1)^2 + (-2)^2, \\ & (-1)^2 + (2)^2, \\ & (-1)^2 + (-2)^2. \end{aligned}$$

For $n = 6$, $d = 1, 2, 3, 6$

$$r_2(6) = 4(1-1) = 0.$$

For $n = 7$, $d = 1, 7$

$$r_2(7) = 4(1-1) = 0.$$

For $n = 8$, $d = 1, 2, 4, 8$

$$\begin{aligned} r_2(8) = 4(1-0) = 4 \Longrightarrow & (2)^2 + (2)^2, \\ & (-2)^2 + (2)^2, \\ & (2)^2 + (-2)^2, \\ & (-2)^2 + (-2)^2. \end{aligned}$$

For $n = 9$, $d = 1, 3, 9$

$$\begin{aligned} r_2(9) = 4(2-1) = 4 \Longrightarrow & (3)^2 + (0)^2, \\ & (-3)^2 + (0)^2, \\ & (0)^2 + (3)^2, \\ & (0)^2 + (-3)^2. \end{aligned}$$

For $n = 10$, $d = 1, 2, 5, 10$

$$\begin{aligned} r_2(10) = 4(2-0) = 8 \Longrightarrow & (3)^2 + (1)^2, \\ & (-3)^2 + (1)^2, \\ & (3)^2 + (-1)^2, \\ & (-3)^2 + (-1)^2, \\ & (1)^2 + (3)^2, \\ & (1)^2 + (-3)^2, \\ & (-1)^2 + (3)^2, \\ & (-1)^2 + (-3)^2. \end{aligned}$$

4 MATLAB Program for Sums of Two Squares

For example $n = 200$ is considered.

```
n=200;
k=1:n;
d=k(rem(n,k)==0);
p=length(d);
z1=0;
for i=1:p
if mod(d(i),4)==1
z1=z1+1;
end
end
z2=0;
for i=1:p
if mod(d(i),4)==3
z2=z2+1;
end
end
z=4*(z1-z2)
```

Output:

```
d = 1   2   4   5   8  10  20  25  40  50 100  200
z = 12
```

5 Conclusion

Sums of squares is one of the interesting topics in the field of number theory. During the survey, we came across identities and formulae for the number of representations of a given positive integer into sums of two squares. Determination of this number is based on divisor function. MATLAB program for the calculation of the same is written which gives us divisors and number of representations of a given positive interger.

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A Note on Onset of Benard – Marangoni Ferroconvection with Basic Equations

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Absract: *A review has been done on an Onset of Benard - Marangoni ferroconvection where the lower rigid surface and the upper horizontal free boundary is open to the atmosphere and are considered to be perfectly insulated to temperature perturbation by considering the various factors such as presence of magnetic field viscosity, temperature dependent viscosity, internal heat generation and the effect of coriolis force in a rotating ferrofluid layer with Magnetic Field Dependent viscosity. The study reveals that presence of the above parameter by considering the combined buoyancy and surface tension forces stabilizes or destabilizes the system by considering the critical values of various parameters that is to hasten the onset of ferroconvection.*

Keywords: *Ferrofluid, buoyancy , Surface tension , Coriolis force, Biot number, non linearity of fluid magnetization , Magnetic number.*

Subject Classification Code:

1 Introduction

Convection is known to be heat transfer method in fluid. Convection in a ferrofluid in the presence of an external magnetic field is said to be ferroconvection. Ferrofluids are colloidal liquids made of nano scale ferromagnetic or ferromagnetic particles suspended in a carrier fluid (usually an organic solvent or water). Each tiny particle is thoroughly coated with a surfactant to inhibit clumping. The ferrofluid is a type of functional fluid whose flow and energy transport is controlled by external magnetic field due to its property it as variety of application in various fields. Ferrofluid is used in rotary seals in computer hard drives and other rotating shaft motors. Loudspeakers use ferrofluid to dampen vibrations. In medicine, ferrofluid is used as a contrast agent for magnetic resonance imaging (MRI) and etc [1]. Ferrofluids usually do not retain magnetization in the absence of an externally applied field. The magnetization of ferromagnetic fluids depends on the magnetic field, temperature and the density of the fluid. Any variation in these quantities can induce a change in body force distribution which leads to convection in the presence of magnetic field gradient known as ferroconvection. Convection can be induced if surface tension forces are the function of temperature. In accordance to that if the ferrofluid layer has an upper surface open to atmosphere then the instability is due to the combined effects of the buoyancy as well temperature dependent surface tension forces known as Benard – Marangoni ferroconvection. The Benard – Marangoni convection problems of ferrofluid layer heated from below under various assumptions is studied by many authors. Here we consider the study done by Nanjundappa et al and I. S. Shivakumara on the onset of Benard – Marangoni ferroconvection by considering the various factors such as magnetic field dependent viscosity [2], internal heat generation [3], temperature dependent viscosity [4] and effect of coriolis force in a rotating ferrofluid layer with magnetic field dependent viscosity [5].

2 Formulation and Analysis

2.1 Benard ferroconvection with magnetic field dependent viscosity

The intent of the paper was to study coupled Benard– Marangoni ferroconvection by considering a Boussinesq ferrofluid layer of thickness d with no lateral boundaries and a uniform vertical magnetic field H_0 with magnetic field dependent viscosity. The lower boundary is rigid with fixed temperature T_0, while the upper non-deformable free boundary is subjected to temperature T_1, surface tension forces and a general thermal boundary condition on the perturbation temperature is imposed. A Cartesian co-ordinate system (x, y, z) is used with the origin at the lower boundary and the z-axis vertically upward. Gravity acts in the negative z-direction, $g = -g\hat{k}$, where $\hat{k}$ is the unit vector in the z-direction. The surface tension σ is assumed to vary linearly with temperature in the form,

$$\sigma = \sigma_0 - \sigma_T(T - T_0) \quad (1)$$

Here σ_0 is the unperturbed value and σ_T is the rate of change of surface tension with temperature. The fluid density ρ is assumed to vary linearly with temperature in the form ρ is,

$$\rho = \rho_0[1 - \alpha t(T - T_0)] \quad (2)$$

where αt is the thermal expansion coefficient and ρ_0 is the density at $T = T_0$. In the study of ferroconvection, we have to solve the Maxwell equations simultaneously with the balance of mass, linear momentum and energy. Since the fluid is assume to be electrically not conducting, the Maxwell equations reduce to,

$$\bigtriangledown \cdot \vec{B} = 0 \quad (3)$$

$$\bigtriangledown \times \vec{H} = 0 \quad (4)$$

where $\vec{B}$ is the magnetic induction and $\vec{H}$ the intensity of magnetic field. In view of equation (4) we can express the magnetic field by a scalar potential as,

$$\vec{H} = \bigtriangledown \varphi \quad (5)$$

Further $\vec{B}$, $\vec{M}$ and $\vec{H}$ are related by,

$$\vec{B} = \mu_0(\vec{M} + \vec{H}) \quad (6)$$

where $\vec{M}$ is the magnetization and the μ_0 magnetic permeability of vacuum. We assume that the magnetization is aligned with the magnetic field, but allow dependence on the magnitude of magnetic field as well as on the temperature in the form [6],

$$\vec{M} = [M_0 + \chi(H - H_0) - K(T - T_0)] \left(\frac{\vec{H}}{H}\right) \quad (7)$$

where

$$M_0 = M(H_0, T_0)$$

$$H = |\vec{H}|$$

$$M = |\vec{M}|$$

$$\chi = \left(\frac{\partial M}{\partial H}\right)_{H_0, T_0} \text{ is the magnetic susceptibility}$$

$$K = -\left(\frac{\partial M}{\partial T}\right)_{H_0,T_0} \quad \text{is the pyromagnetic coeffecient}$$

The momentum equation is given by

$$\rho_0\left[\frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\bigtriangledown)\vec{q}\right] = -\bigtriangledown p + \rho\vec{g} + \mu_0(\vec{M}\cdot\bigtriangledown)\vec{H} + 2\bigtriangledown\cdot[\eta D] \tag{8}$$

where $\vec{q} = (u, v, w)$ is the velocity, p the pressure, t the time and $D = [\bigtriangledown\vec{q} + (\bigtriangledown\vec{q})^T]/2$ the rate of strain tensor. The fluid is assumed to be incompressible having variable viscosity. Experimentally, it has been demonstrated that the magnetic viscosity has got exponential variation, with respect to magnetic field [7]. As a first approximation, for small field variation, linear variation of magnetic viscosity has been used in the form $\eta = \eta_0(1 + \vec{\delta}\cdot\vec{B})$ where $\vec{\delta}$ is the variation coefficient of magnetic field dependent viscosity and is considered to be isotropic, η_0 is taken as viscosity of the fluid when the applied magnetic field is absent [8].

Neglecting viscous dissipation, the energy equation is,

$$\left[\rho_0 C_{V,H} - \mu_0\vec{H}\cdot\left(\frac{\partial\vec{M}}{\partial T}\right)_{V,H}\right]\frac{DT}{Dt} + \mu_0 T\left(\frac{\partial\vec{M}}{\partial T}\right)_{V,H}\frac{D\vec{H}}{Dt} = K_t\bigtriangledown^2 T \tag{9}$$

where, $C_{V,H}$ is the specific heat capacity at constant volume and magnetic field per unit mass, and K_t the thermal conductivity.

The Continuity equation is,

$$\bigtriangledown\cdot\vec{q} = 0 \tag{10}$$

We follow the stability analysis as outlined in the work of [6]. The basic state is quiescent and is given by,

$$\begin{aligned}
\vec{q} &= 0\\
p &= p_b(Z)\\
T_b &= T_0 - \beta Z\\
\vec{H}_b &= \left[H_0 - \frac{K\beta Z}{(1+\chi)}\right]\hat{k}\\
\vec{M}_b &= \left[M_0 + \frac{K\beta Z}{(1+\chi)}\right]\hat{k} \text{ where } \beta = \triangle T/d
\end{aligned}$$

To study the stability of the system, we perturb all the variables in the form,

$$\begin{aligned}
\vec{q} &= \vec{q'}\\
p &= p_b(z) + p'\\
\eta &= \eta_b(z) + \eta'\\
T &= T_b + T'\\
\vec{H} &= \vec{H}_b(z) + H'\\
\vec{M} &= \vec{M}_b(Z) + M'
\end{aligned} \tag{11}$$

where, $\vec{q}$, p', η', T', $\vec{H}$ and $\vec{M}$ are perturbed variables and are assumed to be small.

By substituting equation (11) into equation (3) using equations (6) and (7) and assuming that $k\beta d << (1+\chi)H_0$ as propounded after dropping the primes we obtain [6],

$$\begin{aligned} H_x + M_x &= (1 + M_0/H_0)H_x, \\ H_y + M_y &= (1 + M_0/H_0)H_y, \\ H_z + M_z &= (1 + \chi)H_z - KT \end{aligned} \tag{12}$$

where (H_x, H_y, H_z) and (M_x, M_y, M_z) are (x, y, z) components of perturbed magnetic field and magnetization, respectively.

Substituting equation (11) into equation (8) and linearizing. After neglecting the primes we obtain in components,

$$\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \eta_0[1 + \mu_0\delta(M_0 + H_0) \bigtriangledown^2 u] + \mu_0(M_0 + H_0)\frac{\partial H_x}{\partial z} \tag{13}$$

$$\rho_0 \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} + \eta_0[1 + \mu_0\delta(M_0 + H_0) \bigtriangledown^2 v] + \mu_0(M_0 + H_0)\frac{\partial H_y}{\partial z} \tag{14}$$

$$\rho_0 \frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} + \rho_0\alpha_t gT + \eta_0[1 + \mu_0\delta(M_0 + H_0)\bigtriangledown^2 w] + \mu_0(M_0 + H_0)\frac{\partial H_z}{\partial z} - \mu_0 K\beta H_z + \frac{\mu_0 K^2\beta T}{1+\chi} \tag{15}$$

Partially differentiating equation (13) and (14) with respect to x and y, respectively and by adding them we obtain,

$$\bigtriangledown_1^2 p = -\rho_0\alpha_t qT\frac{\partial T}{\partial z} + \mu_0(M_0 + H_0)\frac{\partial}{\partial z}(\bigtriangledown \cdot \vec{H}) - \mu_0 K\beta\frac{\partial H_z}{\partial z} + \frac{\mu_0 K^2\beta T}{1+\chi}\frac{\partial T}{\partial z} \tag{16}$$

where $\bigtriangledown_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the horizontal Laplacian operator.

Eliminating the pressure term from equation (15) using equation (16) we obtain,

$$\rho_0\frac{\partial}{\partial t} - \eta_0[1 + \mu_0\delta(M_0 + H_0)\bigtriangledown^2]\bigtriangledown^2 w = -\rho\alpha_t q\bigtriangledown_1^2 T + \mu_0 K\beta\frac{\partial}{\partial z}(\bigtriangledown_1^2\varphi) + \frac{\mu_0 K^2\beta T}{1+\chi}(\bigtriangledown_1^2 T) \tag{17}$$

where $\bigtriangledown_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Substituting equation (11) into equation (9) and linearising. After neglecting primes we obtain,

$$\rho_0 C_0\frac{\partial T}{\partial t} - \mu_0 K T_0\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial z}\right) = \rho_0 C_0 - \frac{(\mu_0 K^2 T_0)}{1+\chi}w\beta + K_t \bigtriangledown^2 T \tag{18}$$

where $\rho_0 C_0 = \rho_0 C_{V,H} + \mu_0 K H_0$

Finally equation (3), (4) after using (11) and (12) after neglecting primes yield,

$$\left(1 + \frac{M_0}{H_0}\right)\bigtriangledown_1^2\varphi + (1+\chi)\frac{\partial^2\phi}{\partial z^2} - K\frac{\partial T}{\partial z} = 0 \tag{19}$$

Since the principle of exchange of stability is valid, we assume the normal mode solution in the form,

$$(w, T, \varphi) = (w, \Theta, \phi)(z)\exp^{i(lx+my)} \tag{20}$$

where l and m are wave numbers in the x and y directions respectively. Equation (20) in (17), (18) and (19) and non- dimensionalizing the quantities in the form,

$$(x^*, y^*, z^*) = \left(\frac{x}{d}, \frac{y}{d}, \frac{z}{d}\right)$$
$$W^* = \frac{d}{v}W$$
$$t^* = \frac{v}{d^2}t$$
$$\Theta^* = \frac{k}{\beta\nu d}\Theta$$
$$\phi^* = \frac{(1+\chi)k}{K\beta\nu d^2}\phi$$

$\therefore$ We get,

$$(1+\Lambda)(D^2 - a^2)W = (Ra + R_m)a^2\Theta - a^2 R_m D\phi \tag{21}$$
$$(D^2 - a^2)\Theta = -w \tag{22}$$
$$(D^2 - a^2 M_3)\phi - D\Theta = 0 \tag{23}$$

where

$D = \frac{d}{dz}$ is the differential operator

$a = \sqrt{l^2 + m^2}$ is the overall horizontal wave number

$Ra = \frac{\alpha_t g\beta d^4}{k\nu}$ is the thermal Rayleigh number

$R_m = RaM_1 = \frac{\mu_0 K^2 \beta^2 d^4}{(1+\chi)k\mu}$ the magnetic Rayleigh number

$\Lambda = \delta\mu_0(M_0 + H_0)$ the non-dimensional magnetic feild dependent viscosity parameter

$M_1 = \frac{\mu_0 K^2 \beta}{(1+\chi)\alpha_t \rho_0 g}$ the magnetic number

$M_3 = \frac{(1 + \frac{M_0}{H_0})}{(1+\chi)}$ the measure of non-linearity of magnetizatiopn parameter

$M_2 = \frac{\mu_0 T_0 K^2}{\rho_0 C_0 (1+\chi)}$ the non-dimensional parameter and its value for different carrier liquids turns out to be the order of $10^n eg6$ and hence its effect is neglected as compared to unity

The above boundary equations are to be solved subject to appropriate boundary conditions. The boundary considered are,

$$W = DW = \Theta = \Phi = 0 \text{ at } z = 0 \tag{24}$$

$$W = (1+\Lambda)D^2W + M_a a^2\Theta = D\Theta + Bi\Theta + D\Phi = 0 \text{ at } z = 1 \tag{25}$$

where $Ma = \frac{\sigma_T \Delta T d}{\mu k}$ the Marangoni number and $Bi = \frac{hd}{k_t}$ is the Biot number. The case $Bi = 0$ and $Bi = \infty$ respectively,correspond to constant heat flux and isothermal conditions at the upper boundary. The above equations are solved by employing Rayleigh-Ritz technique with chebyshev polynomials of second kind.

2.2 Onset of Benard - Marangoni Ferroconvection with Internal Heat Generation

The study reveals that by considering an electrically non - conducting Boussinesq ferrofluid layer of thickness 'd' with a uniformly distributed volumetric heat generation. A uniform magnetic field H_0 is applied in the direction normal to the boundaries of the ferrofluid layer. The lower boundary is rigid with fixed temperature T_0 while the upper non-deformable free boundaries are kept at T_l and $T_u(< T_l)$ respectively . A Cartesian co-ordinate system (x, y, z) is used with the origin at the lower boundary and the $z-$axis vertically upward. Gravity acts in the negative $z-$direction,$g = -g\hat{k}$, where $\hat{k}$ is the unit vector in the z-direction. The surface tension σ is assumed to vary linearly with temperature in the form,

$$\sigma = \sigma_0 - \sigma_T(T - T_0) \tag{26}$$

where σ_0 is the unperturbed value and $-\sigma_T$ is the rate of change of surface tension with temperature T. The fluid density ρ is assumed to vary linearly with temperature in the form,

$$\rho = \rho_0[1 - \alpha_t(T - T_0)] \tag{27}$$

where α_t is the thermal expansion coefficient and ρ_0 is the density at $T = T_0$. The governing equations, in the Boussinesq approximation are [9],

$$\bigtriangledown \cdot \vec{V} = 0 \tag{28}$$

$$\rho_0 \left[\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \bigtriangledown)\vec{V}\right] = -\bigtriangledown p + \rho\vec{g} + \mu_0(\vec{M} \cdot \bigtriangledown)\vec{H} + \mu \bigtriangledown^2 \vec{V} \tag{29}$$

$$K_t \bigtriangledown^2 T + Q = \left[\rho_0 C_{V,H} - \mu_0 \vec{H} \cdot \left(\frac{\partial \vec{M}}{\partial T}\right)_{V,H}\right] \times \frac{DT}{Dt} + \mu_0 T \cdot \left(\frac{\partial \vec{M}}{\partial T}\right)_{V,H} \frac{D\vec{H}}{Dt} \tag{30}$$

$$\bigtriangledown \cdot \vec{B} = 0 \tag{31}$$

$$\bigtriangledown \times \vec{H} = 0 \tag{32}$$

$$\vec{B} = \mu_0(\vec{M} + \vec{H}) \tag{33}$$

where $\vec{M}$ is the magnetization and the μ_0 magnetic permeability of vacuum. Following Finlayson, we assume that the magnetization is aligned with the magnetic field, but allow dependence on the magnitude of magnetic field as well as on the temperature in the form,

$$\vec{M} = [M_0 + \chi(H - H_0) - K(T - T_0)]\left(\frac{\vec{H}}{H}\right) \tag{34}$$

where,

$$M_0 = M(H_0, T_0)$$

$$H = |\vec{H}|$$

$$M = |\vec{M}|$$

$$\chi = \left(\frac{\partial M}{\partial H}\right)_{H_0,T_0} \text{ is the magnetic susceptibility}$$

$$K = -\left(\frac{\partial M}{\partial T}\right)_{H_0,T_0} \text{ is the pyromagnetic coeffecient}$$
$$\vec{q} = (u, v, w) \text{ is the velocity vector}$$
$$\bigtriangledown^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \text{ the laplacian operator}$$

And t the time, p the pressure, $\vec{H}$ the magnetic field intensity, $\vec{M}$ is the magnetization, $\vec{B}$ is the magnetic induction, $C_{V,H}$ is the specific heat capacity at constant volume and magnetic field per unit mass, Q is the uniformly distributed volumetric heat generation within ferrofluid layer.

To study the stability of the system, we perturb all the variables in the form,

$$\vec{V} = \vec{V}',\ p = p_b(z) + p',\ T = T_b + T',\ \vec{H}_b(z) + H',\ \vec{M} = \vec{M}_b(Z) + M' \tag{35}$$

Solving the above equations and dropping the primes we get,

$$\begin{aligned} H_x + M_x &= (1 + M_0/H_0)H_x \\ H_y + M_y &= (1 + M_0/H_0)H_y \\ H_z + M_z &= (1 + \chi)H_z - KT \end{aligned} \tag{36}$$

where (H_x, H_y, H_z) and (M_x, M_y, M_z) are (x, y, z) components of perturbed magnetic field and magnetization, respectively. By linearising, eliminating the pressure by operating the curl twice and retaining the z- component of resulting equation we obtain,

$$\begin{aligned} \left(\rho_0\frac{\partial}{\partial t} - \mu\bigtriangledown^2\right)\bigtriangledown^2 W = -\mu_0 K\left(-\frac{Qz}{k_1} + \frac{Qd}{2k_1} - \beta\right) \times \frac{\partial}{\partial t}(\bigtriangledown_h^2\varphi) + \rho_0\alpha_t g\bigtriangledown_h^2 T \\ + \frac{\mu_0 K^2}{1+\chi}\left(-\frac{Qz}{k_1} + \frac{Qd}{2k_1} - \beta\right)\bigtriangledown_h^2 T \end{aligned} \tag{37}$$

$$\frac{\partial T}{\partial t} - \mu_0 T_0 K\frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial z}\right) = k_1\bigtriangledown^2 T + \left[1 - \frac{\mu_0 T_0 K^2}{1+\chi}\right] \times \left(-\frac{Qz}{k_1} + \frac{Qd}{2k_1} - \beta\right) \tag{38}$$

$$\left(1 + \frac{M_0}{H_0}\right)\bigtriangledown_1^2\varphi + (1+\chi)\frac{\partial^2\phi}{\partial z^2} - K\frac{\partial T}{\partial z} = 0 \tag{39}$$

The normal mode expansion of the dependent variables is assumed to be of the form,

$$(w, T, \varphi) = (w, \Theta, \phi)(z)\exp^{[i(lx+my)+\alpha t]} \tag{40}$$

where l and m are wave numbers in the x and y directions respectively, and σ is the growth rate which is complex. Thus by non-dimensionaling the quantities in the form,

$$\begin{aligned} (x^*, y^*, z^*) &= \left(\frac{x}{d}, \frac{y}{d}, \frac{z}{d}\right) \\ W^* &= \frac{d}{v}W \\ t^* &= \frac{v}{d^2}t \\ \Theta^* &= \frac{k}{\beta\nu d}\Theta \end{aligned}$$

$$\Phi^* = \frac{(1+\chi)k}{K\beta\nu d^2}\Phi$$

$\therefore$ We get,

$$[(D^2-a^2)-\sigma](D^2-a^2)W = -R_m a^2[N_s(1-2z)-1]\times(D\phi-\Theta)+R_t a^2\Theta \tag{41}$$

$$(D^2-a^2-Pr\sigma)\Theta - PrM_2\sigma\phi = [N_s(1-2z)-1]\times(1-M_2)W \tag{42}$$

$$(D^2-M_3a^2)\phi = D\Theta \tag{43}$$

where the above equations reduces to,

$$(D^2-a^2)^2W = -R_m a^2[N_s(1-2z)-1]\times(D\phi-\Theta)+R_t a^2\Theta \tag{44}$$

$$(D^2-a^2)\Theta = [N_s(1-2z)-1]W \tag{45}$$

$$(D^2-M_3a^2)\phi = D\Theta \tag{46}$$

The above boundary equations are to be solved subject to appropriate boundary conditions. The boundary considered are,

$$W = DW = \Theta = \Phi = 0 \text{ at } z=0 \tag{47}$$

$$W = D^2W + M_a a^2\Theta = D\Theta + D\Phi = 0 \text{ at } z=1 \tag{48}$$

The above equations are solved by employing the Galerkin technique and regular perturbation method and observed the values obtained by both methods complement each other.

2.3 Onset of Benard - Marangoni Ferroconvection with Temperature Dependent Viscosity

In the study we consider a horizontal ferrofluid layer of thickness 'd' with an uniform magnetic field H_0 in the vertical direction . The lower boundary is rigid with fixed temperature T_l, while the upper non - deformable free boundaries are kept at $T_u(< T_l)$ respectively. A Cartesian coordinate system (x, y, z) is used with the origin at the lower boundary and the $z-$ axis vertically upward [10]. The surface tension and density of a ferrofluid is considered to be relatively high at a adjacent phase. the surface tension σ is assumed to vary linearly with temperature in the form,

$$\sigma = \sigma_0 - \sigma_T(T-T_0) \tag{49}$$

where σ_0 is the unperturbed value and $-\sigma_T$ is the rate of change of surface tension with temperature T. whereas the viscosity η of the ferrofluid is assumed to vary exponentially with temperature in the form,

$$\eta = \eta_0 \exp[-\gamma(T-T_r)] \tag{50}$$

where η_0 is the reference value at the reference temperature T_r and γ is a positive constant.

The relevant governing equations are,

$$\bigtriangledown\cdot\vec{q} = 0 \tag{51}$$

$$\rho_0\left[\frac{\partial\vec{q}}{\partial t}+(\vec{q}\cdot\bigtriangledown)\vec{q}\right] = -\bigtriangledown p + \bigtriangledown\cdot[\eta(\bigtriangledown\vec{q}+\bigtriangledown\vec{q^T})] + \mu_0(\vec{M}\cdot\bigtriangledown)\vec{H} \tag{52}$$

$$K_t \bigtriangledown^2 T = \left[\rho_0 C_{V,H} - \mu_0 \vec{H} \cdot \left(\frac{\partial \vec{M}}{\partial T}\right)_{V,H}\right] \frac{DT}{Dt} + \mu_0 T \left(\frac{\partial \vec{M}}{\partial T}\right)_{V,H} \cdot \frac{D\vec{H}}{Dt} \tag{53}$$

$$\bigtriangledown \cdot \vec{B} = 0 \tag{54}$$

$$\bigtriangledown \times \vec{H} = 0 \tag{55}$$

$$\vec{B} = \mu_0(\vec{M} + \vec{H}) \tag{56}$$

$$\vec{M} = [M_0 + \chi(H - H_0) - K(T - T_0)] \left(\frac{\vec{H}}{H}\right) \tag{57}$$

where,

$$M_0 = M(H_0, T_0)$$
$$H = |\vec{H}|$$
$$M = |\vec{M}|$$
$$\chi = \left(\frac{\partial M}{\partial H}\right)_{H_0,T_0} \text{ is the magnetic susceptibility}$$
$$K = -\left(\frac{\partial M}{\partial T}\right)_{H_0,T_0} \text{ is the pyromagnetic coeffecient}$$
$$\vec{q} = (u, v, w) \text{ is the velocity vector}$$
$$\bigtriangledown^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \text{ the laplacian operator}$$

and t denotes time, p the pressure, $\vec{H}$ the magnetic field intensity, $\vec{M}$ is the magnetization, $\vec{B}$ is the magnetic induction, $C_{V,H}$ is the specific heat capacity at constant volume and magnetic field per unit mass.

To study the stability of the system, we perturb all the variables in the form,

$$\vec{V} = \vec{V}', p = p_b(z) + p', T = T_b + T', \vec{H}_b(z) + H', \vec{M} = \vec{M}_b(Z) + M', \eta = \eta_b(z) + \eta' \tag{58}$$

Then,

$$\eta = \eta_0 \exp\left[\gamma\beta\left(z - \frac{d}{2}\right) + \gamma(T_r - T_\alpha) - \gamma T'\right] \tag{59}$$

Solving the above equations after dropping the primes we get,

$$\begin{aligned} H_x + M_x &= (1 + M_0/H_0)H_x \\ H_y + M_y &= (1 + M_0/H_0)H_y \\ H_z + M_z &= (1 + \chi)H_z - KT \end{aligned} \tag{60}$$

where (H_x, H_y, H_z) and (M_x, M_y, M_z) are (x, y, z) components of perturbed magnetic field and magnetization, respectively. By eliminating the pressure term by operating the curl twice and linearising we obtain,

$$\left(\rho_0 \frac{\partial}{\partial t}\right) \bigtriangledown^2 w = \eta(z) \bigtriangledown^4 w + 2\frac{\partial \eta(z)}{\partial z} \bigtriangledown^2 \left(\frac{\partial w}{\partial z}\right) + \frac{\partial^2 \eta(z)}{\partial z^2} \times (\bigtriangledown^2 w - 2 \bigtriangledown_h^2 w) + \frac{\mu_0 k\beta}{1+\chi}\frac{\partial}{\partial z}(\bigtriangledown_h^2 T) + \frac{\mu_0 k^2\beta}{1+\chi}\frac{\partial}{\partial z}(\bigtriangledown_h^2 T) \tag{61}$$

$$\rho_0 C_0 \frac{\partial T}{\partial t} - \mu_0 T_0 K \frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial z}\right) = k_t \bigtriangledown^2 T + \left[\rho_0 C_0 - \frac{\mu_0 T_0 K^2}{1+\chi}\right] w\beta \tag{62}$$

After simplification the above equation reduces to,

$$\left(1 + \frac{M_0}{H_0}\right) \bigtriangledown_n^2 \varphi + (1+\chi)\frac{\partial^2 \varphi}{\partial z^2} - K\frac{\partial T}{\partial z} = 0 \tag{63}$$

The normal mode expansion of the dependent variables is assumed to be of the form,

$$(w, T, \varphi) = (w, \Theta, \phi)(z) \exp^{i(lx+my)} \tag{64}$$

where l and m are wave numbers in the x and y directions respectively and Non-dimensionalizing the above quantities we have,

$$\begin{aligned}
(x^*, y^*, z^*) &= \left(\frac{x}{d}, \frac{y}{d}, \frac{z}{d}\right) \\
W^* &= \frac{d}{v} W \\
\Theta^* &= \frac{k}{\beta \nu d}\Theta \\
\phi^* &= \frac{(1+\chi)k}{K\beta\nu d^2}\phi \\
f(\bar{z}) &= \frac{\eta(z)}{\eta_0}
\end{aligned}$$

$\therefore$ We get,

$$\bar{f}((D^2-a^2)^2)W + 2D\bar{f}(D^2-a^2)DW + D^2\bar{f}(D^2+a^2)W = -R_m a^2(D\phi - \Theta) \tag{65}$$

$$(D^2 - a^2)\Theta = -(1 - M_2)W \tag{66}$$

$$(D^2 - M_3 a^2)\phi = D\Theta \tag{67}$$

On simplification,

$$f(\bar{z}) = \exp\left[B\left(z - \frac{1}{2}\right) + \frac{(T_r - T_\alpha)}{\beta d}\right] \tag{68}$$

where $B = \gamma\beta d$ is the dimensionless viscosity parameter . If the reference temperature $T_r = T_\alpha$ then,

$$f(\bar{z}) = \exp\left[B\left(z - \frac{1}{2}\right)\right] \tag{69}$$

subjected to boundary condition

$$W = DW = \Theta = \Phi = 0 \text{ at } z = 0 \tag{70}$$

$$W = \bar{f}D^2W + M_a a^2\Theta = D\Theta + D\Phi = 0; \text{at } z = 1 \tag{71}$$

The above equations are solved by employing the Galerkin technique and regular perturbation method and observed the values obtained by both methods complement each other.

2.4 Effect of Coriolis Force on Benard - Marangoni Convection in a Rotating Ferrofluid Layer with MFD Viscosity

The intent of the paper was to study coupled Benard– Marangoni ferroconvection by considering a Boussinesq ferrofluid layer of thickness d permeated by uniform applied magnetic field H_0 acting in the vertical direction . Which is bounded below by rigid surface and above by a non - deformable free surface. where the layer is rotating uniformly about its axis with an angular velocity $\vec{\Omega} = \Omega\hat{k}$. A cartesian coordinate system (x, y, z) is used with the origin at the lower boundary and the $z-$ axis vertically upwards.The surface tension σ is assumed to vary linearly with temperature in the form,

$$\sigma = \sigma_0 - \sigma_T \Delta T \tag{72}$$

where σ_0 is the unperturbed value and $-\sigma_T$ is the rate of change of surface tension with temperature T.

A linear variation in the viscosity with respect to magnetic field is of the form [8],

$$\eta = \eta_0(1 + \vec{\delta} \cdot \vec{B}) \tag{73}$$

where η_0 is the viscosity of the fluid in the absence of magnetic field and $\vec{\delta}$ is the coefficient of magnetic field dependent viscosity. The relevant governing equations are,

$$\bigtriangledown \cdot \vec{V} = 0 \tag{74}$$

$$\rho_0 \left[\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \bigtriangledown)\vec{V} \right] = -\bigtriangledown P + \rho_0[1 - \alpha_t(T - \bar{T})]\vec{g} + 2 \bigtriangledown \cdot [\eta D] + \mu_0(\vec{M} \cdot \bigtriangledown)\vec{H} + 2\rho_0(\vec{V} \times \vec{\Omega}) \tag{75}$$

with usual notation and last term represent the Coriolis force.

Considering the energy equations which obeys the Fourier's laws and ignoring the primes we get [6],

$$\left(\rho_0 \frac{\partial}{\partial t} - \eta \nabla^2 \right) \bigtriangledown^2 w = \rho_0 \alpha_t g \bigtriangledown_1^2 T - 2\rho_0 \Omega \frac{\partial \xi}{\partial z} - \mu_0 K \beta \frac{\partial}{\partial z}(\bigtriangledown_1^2 \varphi) + \frac{\mu_0 K^2 \beta}{1 + \chi}(\bigtriangledown_1^2 T) \tag{76}$$

where $\eta = \eta_0[1 + \delta\mu_0(M_0 + H_0)]$ and $\xi = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$ is the z- component of the vorticity arising due to rotation and an equation is obtained for ξ

$$\rho_0 \frac{\partial \xi}{\partial t} = \eta \bigtriangledown^2 \xi + 2\rho_0 \Omega \frac{\partial w}{\partial z} \tag{77}$$

linearisation the above equations,

$$\rho_0 C_0 \frac{\partial T}{\partial t} - \mu_0 T_0 K \frac{\partial}{\partial t} \left(\frac{\partial \varphi}{\partial z} \right) = k_t \bigtriangledown^2 T + \left[\rho_0 C_0 - \frac{\mu_0 T_0 K^2}{1 + \chi} \right] w\beta \tag{78}$$

leads to

$$\left(1 + \frac{M_0}{H_0} \right) \bigtriangledown_1^2 \varphi + (1 + \chi) \frac{\partial^2 \varphi}{\partial z^2} - K \frac{\partial T}{\partial z} = 0 \tag{79}$$

The normal mode hypothesis and expanded in the form

$$f(x, y, z, t) = f(z, t) \exp^{i(lx+my)} \tag{80}$$

where l and m are wave numbers in the x and y directions respectively,

$$\left[\rho_0 \frac{\partial}{\partial t} - \eta(D^2 - a^2)\right](D^2 - a^2)w = -a^2 \alpha_t g \Theta + a^2 \mu_0 K \beta D\varphi - \frac{a^2 \mu_0 K^2 \beta}{1+\chi}\Theta - 2\rho_0 \Omega D\xi \tag{81}$$

$$\rho_0 \frac{\partial \xi}{\partial t} = \eta(D^2 - a^2)\xi + 2\rho_0 \Omega Dw \tag{82}$$

$$\left(1 - \frac{\mu_0 K^2 T_0}{(1+\chi)\rho_0 C_0}\right)\omega\beta = \frac{\partial \Theta}{\partial t} - K(D^2 - a^2)\Theta - \frac{\mu_0 K T_0}{\rho_0 C_0}\frac{\partial}{\partial t}(D\varphi) \tag{83}$$

$$0 = (1+\chi)^2 \varphi - \left(1 + \frac{M_0}{H_0}\right) a^2 \varphi - KD\Theta \tag{84}$$

Simplified by dimensionless variables as,

$$(x^*, y^*, z^*) = \left(\frac{x}{d}, \frac{y}{d}, \frac{z}{d}\right)$$

$$\omega^* = \frac{d}{v}\omega$$

$$\Theta^* = \frac{k}{\beta \nu d}\Theta$$

$$\Phi^* = \frac{(1+\chi)k}{K\beta\nu d^2}\Phi$$

$$t^* = \frac{\nu t}{d^2}$$

$$\xi^* = \frac{d^2 \xi}{\nu}$$

$\therefore$ We obtain,

$$\left[(1+\Lambda)(D^2 - a^2) - \frac{\partial}{\partial t}\right](D^2 - a^2)\omega = Ta^{\frac{1}{2}} D\xi + R_t a^2 \Theta + R_m a^2 (\Theta - D\varphi) \tag{85}$$

$$\left[(1+\Lambda)(D^2 - a^2) - \frac{\partial}{\partial t}\right]\xi = -Ta^{\frac{1}{2}} D\xi \tag{86}$$

$$(D^2 - a^2 - Pr\frac{\partial}{\partial t}\Theta + PrM_2 \frac{\partial}{\partial t} D\varphi = -(1 - M_2)\omega \tag{87}$$

$$(D^2 - M_3 a^2)\varphi - D\Theta = 0 \tag{88}$$

By reducing we get,

$$\left[(1+\Lambda)(D^2 - a^2) - \frac{\partial}{\partial t}\right](D^2 - a^2)\omega = Ta^{\frac{1}{2}} D\xi + R_t a^2 \Theta + R_m a^2 (D\Phi - \Theta) \tag{89}$$

$$[(1+\Lambda)(D^2 - a^2) - \omega]\xi = -Ta^{\frac{1}{2}} DW \tag{90}$$

$$(D^2 - a^2 - Pr\omega)\Theta = -W \tag{91}$$

$$(D^2 - M_3 a^2)\Phi - D\Theta = 0 \tag{92}$$

solving with the boundary conditions

$$W = DW = D\Theta = \xi = 0 \text{ at } z = 0 \tag{93}$$

$$W = (1+\Lambda)D^2W + M_a a^2\Theta = D\xi = D\Theta + Bi\Theta = 0 \text{ at } z = 1 \tag{94}$$

The above equations are solved by employing the Galerkin technique and regular perturbation method and observed the values obtained by both methods complement each other.

3 Conclusion

1. The study reveals that increase in the value of magnetic field dependent viscosity parameter Λ and Biot number Bi effects in delaying, where as increase in the value of magnetic Rayleigh number R_m and non linearity of fluid magnetization parameter M_3 is to advance the onset of Benard-Marangoni ferroconvection. Further increase in Bi and Λ as well as decrease in M_1 and M_3 values decreases the dimension of the convection cells. Also as $M_3 \longrightarrow \infty$, the results to Benard - Marangoni problem for ordinary fluids.

2. The study reveals that the increased value in magnetic Rayleigh number R_m and the internal heat generation source strength N_s together is to reinforce and hasten the onset of Benard - Marangoni ferroconvection , where as the non-linearity of fluid magnetization parameter M_3 and magnetic number M_1(in the absence of internal heat generation) has no effect.

3. The viscosity parameter B has a dual effect on the system depending upon the strength of the system initially but a reverse once B exceeds certain threshold value. Further the increase in the Marangoni number and magnetic Rayleigh number will reinforce and hasten the onset of ferroconvection.

4. The presence of coriolis force due to rotation is to reduce the intensity of Benard - Marangoni convection in a rotating ferrofluid layer and the effect of increasing the values of Biot number Bi and MFD viscosity parameter Λ is to delay with increasing value of a magnetic parameter M_1 is to advance the onset of Benard - Marangoni convection.

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Homotopy Analysis Method for Nonlinear Boundary Value Problems

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Absract: *Magnetohydrodynamic(MHD) flows have applications in meteorology, solar physics, cosmic fluid dynamics, astrophysics, geophysics and in the motion of earth's core. In addition from the technological point of view, MHD free convection flows have significant applications in the field of stellar and planetary magnetospheres, aeronautical plasma flows, chemical engineering and electronics. The object of the present paper is, "the study of MHD effects on a free convection boundary layer flow past a semi-infinite moving vertical plate embedded in a porous medium". The governing equations are transformed by using similarity transformation and the resultant dimensionless equations are solved numerically using Homotopy Analysis Method. The effects of various governing parameters on the velocity, temperature, concentration, skin-friction coefficient, Nusselt number and Sherwood number are analyzed in detail.*

Keywords: *Homotopy analysis method, Heat and mass transfer, skin-fricrion, Nusselt number, Sherwood number*

Subject Classification Code :

1 Introduction

MHD Free Convection Flow Past a Semi Infinite Moving Vertical Porous Plate Embedded in a Porous Medium

Combined heat and mass transfer (or double-diffusion) in fluid-saturated porous media finds applications in a variety of engineering processes such as heat exchanger devices, petroleum reservoirs, chemical catalytic reactors and processes, geothermal and geophysical engineering, moisture migration in a fibrous insulation and nuclear waste disposal and others. Double diffusive flow is driven by buoyancy due to temperature and concentration gradients. Bejan and Khair [1] investigated the free convection boundary layer flow in a porous medium owing to combined heat and mass transfer. Lai and Kulacki[2] used the series expansion method to investigate coupled heat and mass transfer in natural convection from a sphere in a porous medium. The suction and blowing effects on free convection coupled heat and mass transfer over a vertical plate in a saturated porous medium were studied by Raptis et al.[3] and Lai and Kulacki [4] respectively. Magnetohydrodynamic flows have applications in meteorology, solar physics, cosmic fluid dynamics, astrophysics, geophysics and in the motion of earth's core. In addition from the technological point of view, MHD free convection flows have significant applications in the field of stellar and planetary magnetospheres, aeronautical plasma flows, chemical engineering and electronics. An excellent summary of applications is given by Huges and Young [5]. Soundalgekar et al.[6] analyzed the problem of free convection effects on Stokes problem for a vertical plate under the influence of transversely applied magnetic field with mass transfer. Raptis [7] studied mathematically the case of time varying two dimensional natural convective flow of an incompressible, electrically conducting fluid along an infinite vertical porous plate embedded in a porous medium. Helmy [8] analyzed MHD unsteady free convection flow past a vertical porous plate embedded in a porous medium. Elabashbeshy [9] studied heat and mass transfer along a vertical plate in the presence of magnetic field. Chamkha and Khaled [10] investigated the problem of coupled heat and mass transfer by magnetohydrodynamic free convection from an inclined plate in the presence of internal heat generation or absorption. Kim [11] studied unsteady MHD convective heat transfer past a semi-infinite vertical porous moving plate with variable suction by assuming that the free stream velocity follows the exponentially increasing small perturbation law. Chamkha [12] extended the problem of Kim[11] to heat absorption and mass transfer effects. The object of the present paper is to study the MHD effects on a free convection boundary layer flow past a semi-infinite moving vertical plate embedded in a porous medium. The governing equations are transformed by using similarity transformation and the resultant dimensionless equations are solved numerically using the Runge-Kutta method with shooting technique. The effects of various governing parameters on the velocity, temperature, concentration, skin-friction coefficient, Nusselt number and Sherwood number are shown in figures and tables and analyzed in detail.

2 Mathematical Analysis

A steady two-dimensional hydromagnetic flow of a viscous incompressible, electrically conducting and viscous dissipating fluid past a semi-infinite moving vertical porous plate embedded in a porous medium is considered. The flow is assumed to be in the $x-$ direction, which is taken along the semi-infinite plate and $y-$ axis normal to it. The plate is maintained at a constant temperature T_w, which is higher than the constant temperature T_∞ of the surrounding fluid and a constant concentration C_w, which is greater than the constant concentration C_∞ of the surrounding fluid. A uniform magnetic field is applied in the direction perpendicular to the plate. The fluid is assumed to be slightly conducting, and hence the magnetic Reynolds number is much less than unity and the induced magnetic field is negligible in comparison with the applied magnetic field. It is further assumed that there is no applied voltage, so that electric field is absent. It is also assumed that all the fluid properties are constant except that of the influence of the density variation with temperature and concentration in the body force term (Boussinesq's approximation). Then, under the above assumptions, the governing equations are,

Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

Momentum equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = v\frac{\partial^2 u}{\partial y^2} + g\beta(T - T_\infty) + g\beta^*(C - C_\infty) - \frac{\sigma B_0^2}{\rho}u - \frac{v}{K'}u \tag{2}$$

Energy equation:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2} \tag{3}$$

Species equation:

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D\frac{\partial^2 C}{\partial y^2} \tag{4}$$

The boundary conditions for the velocity, temperature and concentration fields are,

$$\begin{aligned} u = U_0, v = v_0(x), T = T_w, C = C_w \quad at \quad y = 0 \\ u \to 0, v \to 0, T \to T_\infty, C \to C_\infty \quad as \quad y \to \infty \end{aligned} \tag{5}$$

where U_0 is the uniform velocity of the plate and $v_0(x)$ - the suction vecolity at the plate, u,v- the velocity components in directions respectively, ρ- the fluid density, g - the acceleration due to gravity, β and β^* - the permeability of the porous medium, T- the temperature of the fluid in the boundary layer, ν - the kinematic viscosity, σ - the electrical conductivity of the fluid, T_∞ - the temperature of the fluid far away from the plate, α- the thermal diffusivity, C - the species concentration in the boundary layer, C_∞ - the species concentration in the fluid far away from the plate, B_0 - the magnetic induction, k - the thermal conductivity, c_p- the specific heat at constant pressure, and D- the mass diffusivity.

The equations (2)-(4) are nonlinear partial differential equations and hence analytical solution is not possible.Therefore numerical technique is employed to obtain the required solution. Numerical computations are greatly facilitated by non-dimensionalization of the equations. Proceeding with the analysis, we introduce the following similarity transformations and dimensionless variables which will convert the partial differential equations from two independent variables (x, y) to a system of coupled, non-linear ordinary differential equations in a single variable (η) i.e., coordinate normal to the plate.

In order to write the governing equations and the boundary conditions in dimensionless form, the following non-dimensional quantities are introduced.

$$\begin{aligned} \eta = y\sqrt{\frac{U_0}{2vx}}, \psi = \sqrt{vxU_0}f(\eta), \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty}, \\ u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x}, Gr = \frac{g\beta(T_w - T_\infty)x^3}{v^2}, Gm = \frac{g\beta^*(C_w - T_\infty)}{v^2}, \\ M = \frac{2\sigma B_0^2 x}{\rho U_0}, Pr = \frac{vc_p}{k}, K = \frac{vx}{K' U_\infty} \end{aligned} \tag{6}$$

where ψ is the stream function, θ - the non-dimensional temperature function, ϕ - the non-dimensional concentration, Gr - the thermal Grashof number, Gm - the solutal Grashof number, M - the magnetic field parameter, K - the permeability parameter, Pr - the Prandtl number and Sc - the Schmidt number.

The mass conservation equation (1) is satisfied by the Cauchy-Riemann Equations,

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}.$$

In view of the equation (6) and following the analysis of Chamkha and Issa [13], the equations (2), (3) and (4) reduce to the following non-dimensional form,

$$f''' + ff'' + Gr\theta + Gm\phi - (M + K)f' = 0 \tag{7}$$

$$\theta'' + Prf\theta' = 0 \tag{8}$$

$$\phi" + Scf\phi' = 0 \tag{9}$$

The corresponding boundary conditions are,

$$f = f_w, f' = 1, \theta = 1, \phi = 1 \quad at \quad \eta = 0$$
$$f' \to 0, \theta \to 0, \phi \to 0 \quad as \quad \eta \to \infty \tag{10}$$

where f is the dimensionless stream function, $f_w = -\nu_0\sqrt{\frac{2x}{vU_0}}$ is the dimensionless suction velocity and primes denote partial differentiation with respect to the variable η.

The skin-friction coefficient, Nusselt number and Sherwood number are important physical parameters for this type of boundary layer flow. Knowing the velocity field, the skin-friction at the plate can be obtained, which in non-dimensional form is given by,

$$C_f = 2(Re)^{-}\frac{1}{2}f''(0)$$

Knowing the temperature field, the rate of heat transfer coefficient can be obtained, which in non-dimensional form, in terms of the Nusselt number, is given by,

$$Nu = -(Re)^{\frac{1}{2}}f''(0)$$

Knowing the concentration field, the rate of mass transfer coefficient can be obtained, which in non-dimensional form, in terms of the Sherwood number, is given by,

$$Sh = -(Re)^{\frac{1}{2}}\phi'(0)$$

where $Re = \frac{U_0 x}{v}$ is the Reynolds number.

3 Solution of the Problem

The set of coupled non-linear governing boundary layer equations (7) - (9) together with the boundary conditions (10) are solved numerically by using Runge-Kutta fourth order technique along with shooting method. First of all, higher order non-linear differential equations (7)- (9) are converted into simultaneous linear differential equations of first order and they are further transformed into initial value problem by applying the shooting technique (Jain et al. [14]). The resultant initial value problem is solved by employing Runge-Kutta fourth order technique. The step size $\Delta\eta = 0.05$ is used to obtain the numerical solution with five decimal place accuracy as the criterion of convergence. From the process of numerical computation, the skin-friction coefficient, the Nusselt number and the Sherwood number, which are respectively proportional to $f''(0)$, $-\theta'(0)$ and $-\phi"(0)$, are also sorted out and their numerical values are presented in a tabular form.

4 Result and Discussions

1. As a result of the numerical calculations, the dimensionless velocity, temperature and concentration distributions for the flow under consideration are obtained and their behaviour have been discussed for variations in the governing parameters viz., the thermal Grashof number G_r, solutal Grashof number G_m, magnetic field parateter M, permtability parameter K, Prandtl number P_r, Schmidt number G_c and suction parameter f_w.

2. For various values of the magnetic parameter *M*, the velocity profiles are plotted in Figure (a). It can be seen that as M increases, the velocity decreases. This result qualitatively agrees with the expectations, since the magnetic field exerts a retarding force on the free convection flow.

3. The effect of the permeability parameter K on the velocity field is shown in Figure (b). The parameter K is inversely proportional to the actual permeability K' of the porous medium. An increase in K will therefore increase the resistance of the porous medium (as the permeability physically becomes less with increasing (K') which will tend to decelerate the flow and reduce the velocity. This behaviour is evident from Figure (b).

4. The influence of the thermal Grashof number on the velocity is presented in Figure (c). The thermal Grashof number G_r signifies the relative effect of the thermal buoyancy force to the viscous hydrodynamic force in the boundary layer. As expected, it is observed that there is a rise in the velocity due to the enhancement of thermal buoyancy force. Here, the positive values of G_r correspond to cooling of the plate. Also, as G_r increases, the peak values of the velocity increases rapidly near the porous plate and then decays smoothly to the free stream velocity. Figure (d) presents typical velocity profiles in the boundary layer for various values of the solutal Grashof number Gm, while all other parameters are kept at some fixed values. The solutal Grashof number G_m defines the ratio of the species buoyancy force to the viscous hydrodynamic force. As expected, the fluid velocity increases and the peak value is more distinctive due to increase in the species buoyancy force. The velocity distribution attains a distinctive maximum value in the vicinity of the plate and then decreases properly to approach the free stream value

5. Figure (e) illustrate the velocity profile for different values of the Prandtl number P_r. The Prandtl number defines the ratio of momentum diffusivity to thermal diffusivity. The numerical results show that the effect of increasing values of Prandtl number results in a decreasing velocity. The reason is that smaller values P_r of are equivalent to increasing the thermal conductivities, and therefore heat is able to diffuse away from the heated plate more rapidly than for higher values of P_r. Hence in the case of smaller Prandtl numbers as the boundary layer is thicker and the rate of heat transfer is reduced.

6. The influence of the Schmidt number $S_c$on the velocity profiles are plotted in Figure (f). The Schmidt number embodies the ratio of the momentum to the mass diffusivity. The Schmidt number therefore quantifies the relative effectiveness of momentum and mass transport by diffusion in the hydrodynamic (velocity) and concentration (species) boundary layers. This causes the concentration buoyancy effects to decrease yielding a reduction in the fluid velocity.

7. Figure (g) illustrates the influence of the suction parameter f_w on the velocity. It is observed that an increase in the suction parameter results in a decrease in the velocity. Figure (h) shows the effect of Prandtl number P_r on the temperature. It is seen that the temperature decreases as the Prandtl number increases. Figure (i) depict the concentration profiles for different values of the Schmidt number S_c. It is noticed that an increase in the Schmidt number Sc results in a decrease in the concentration within the boundary layer.

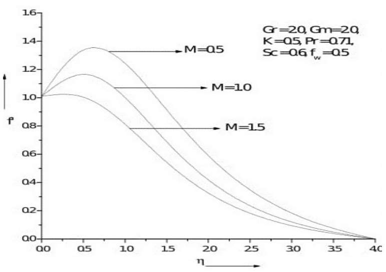
8. The effects of various governing parameters on the skin-friction coefficient C_f , Nusselt number N_u and the Sherwood number S_h are shown in Table 1., it is observed that as G_r

increases, there is a rise in the skin-friction coefficient, Nusselt number and the Sherwood number. As G_m increases the skin-friction coefficient and Sherwood number increase, where as the Nusselt number decreases. As M increases, there is a fall in the skin-friction coefficient and there is a rise in both the Nusselt number and the Sherwood number. As P_r increases there is a fall in the skin-friction coefficient and a rise in the Nusselt number. As S_c increases there is a rise in the Sherwood number.

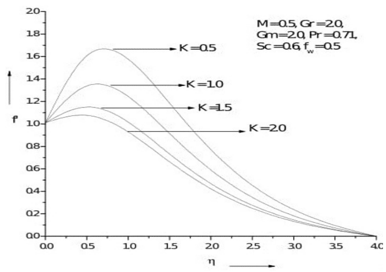
G_r	G_m	M	P_r	S_c	C_ρ	N_u	S_h
2.0	2.0	1.0	0.71	0.6	1.36639	1.82378	0.71743
4.0	2.0	1.0	0.71	0.6	1.93338	2.27858	1.07356
2.0	4.0	1.0	0.71	0.6	2.16667	1.82073	1.25739
2.0	2.0	2.0	0.71	0.6	1.25874	2.88405	1.46477
2.0	2.0	2.0	1.0	0.6	0.68509	3.45210	1.46477
2.0	2.0	2.0	1.0	0.78	0.68509	3.45210	1.84218

Table 1: Numerical values of the skin-friction coefficient, Nauseate and Sherwood numbers

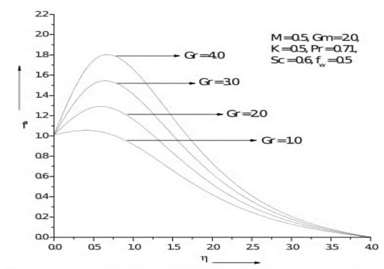
Graphs



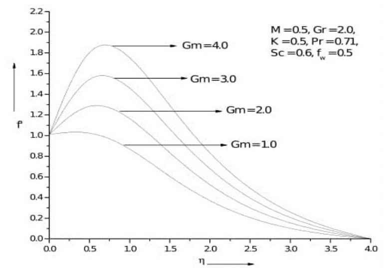
(a) Velocity profile for different values of M



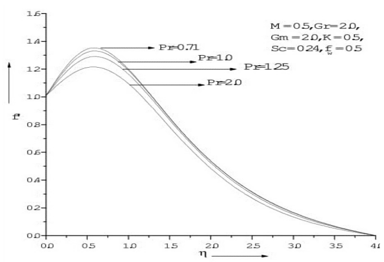
(b) Velocity profile for different values of K



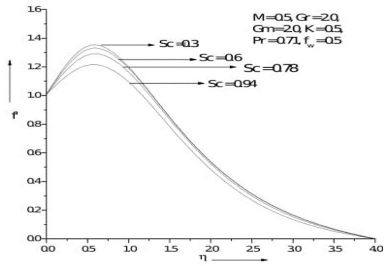
(c) Velocity profile for different values of G_r



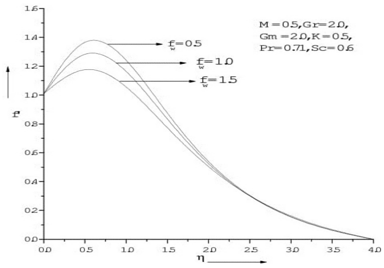
(d) Velocity profile for different values of G_m



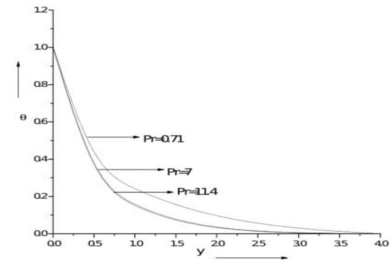
(e) Velocity profile for different values of P_r



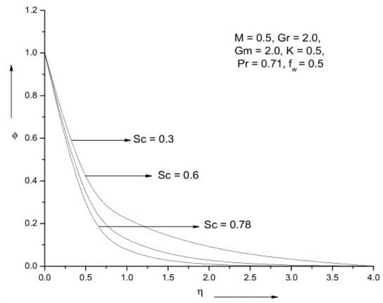
(f) Velocity profile for different values of S_c



(g) Velocity profile for different values of F_w



(h) Temperature profile for different values of P_r



(i) Concentration profile for different values of S_c

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Numerical Solution of Nonlinear Boundary Value Problems

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Absract: *In this paper we will focus on the Finite difference method involved in solving systems of nonlinear boundary value problems for ordinary differential equations. Here, we have applied Successive iteration method and Newton method to find the unknowns which is later compared with the exact solution. We will give some applications as well as the advantages and disadvantages of Finite difference method. And we will solve few boundary value problem of a nonlinear ordinary differential equation using finite difference method.*

Keywords: *Finite Difference Method, Nonlinear Boundary value problems, Successive iteration method and Newton method and Truncation error.*

Subject Classification Code : 65N06

1 Introduction

The finite difference approximations for the derivatives are one of the simplest and oldest methods to solve differential equations. It was already known by L. Euler (1707-1783) in the year 1768, in one dimension of space and was probably extended to dimension two by C. Runge (1856-1927) in the year 1908. The advent of finite difference techniques in numerical applications began in the early 1950s and their development was stimulated by the emergence of computers that offered a convenient framework for dealing with complex problems of science and technology.

The principle of finite difference methods is similar to the numerical schemes used to solve ordinary differential equations. The domain is partitioned in space and approximations of the solution are compute at the space. The error between the numerical solution and the exact solution is determined by the error that is committed by going from a differential operator to a difference operator. This error is called the discretization error or truncation error. The term truncation error reflects the fact that a finite part of a Taylor series is used in the approximation.

In Math 3351, Courtney Remani focused on solving nonlinear equations involving only a single variable. They used many methods like Newton's method, the Secant method, and the Bisection method and also they examined numerical methods such as Runge-Kutta method that are used to solve initial-value problems for Ordinary differential equations. They focused only on solving nonlinear equations with only one variable rather than nonlinear equations with several variable. [1]

2 Finite Difference Method

Finite difference methods (FDM) are numerical methods for solving differential equations by approximating them with difference equations, in which finite differences approximate the derivatives. Finite difference method convert a non-linear ordinary differential equations into a system of non-linear equations which can be solved by matrix algebra techniques. It computes the solutions numerically at a predefined set of discrete points in the structured grid of a computational domain. These discrete points along with their inter connections are called nodal points of the grid or mesh. The procedure of identifying the grid points for a given domain is called the discretization of the domain, which is the first step in the finite difference method. [2]

The finite difference method approximates the differential operator by replacing the derivatives in the equation using differential quotients, which involve values of the solution at discrete mesh points in the domain under study. Repeated applications of this representation set up algebraic systems of equations in terms of unknown mesh point values. The method is a classical one, having been established almost a century ago. Timoshenko and Goodier (1970) provided some details on its applications in elasticity. The major difficulty with this scheme lies in the inaccuracies in dealing with regions of complex shape, although this problem can eliminated through the use of coordinate transformation techniques. [3]

Weighted residual methods form a class of methods that can be used to solve differential equations. They make use of approximation functions that are appropriately weighted in order to find a solution which approximates the solution to the differential equations as closely as possible. Weighted residual methods are used in several other commonly encountered methods for solving differential equations numerically. It forms the basis for most of the numerical schemes.

The concept of FDM is focused on approximating differentials. In contrast to this, weighted residual methods evaluate the integral of differential equation and optimize an approximation such that the integrals of the correct and the approximated solutions match on a given domain. Therefore these equations use integral approximations. FDM uses an approximation of the differential of the differential equation. Hence it is a differential approximation. The mathematics of FDM is based on Taylor series approximations. The most common equations are

- Central finite difference scheme((6)), for approximating first derivatives.
- Forward finite difference scheme((4)), for approximating first derivatives.
- Backward finite difference scheme((5)), for approximating first derivatives.
- Central finite difference scheme((7)), for approximating second derivatives.

These schemes are used in many forms in numerical solvers. The difference in the solution i.e. the finite change of the solution is approximated on a very small finite interval using one of these equations. All of these equations are linear i.e. the solution is linearly approximated. Obviously, this approximation is only correct if the interval on which the function is linearized is sufficiently small. Otherwise, the solution becomes inexact [4].

The particular difference quotient and step size h are chosen to maintain a specified order of truncation error. However, h cannot be chosen too small because of the general instability of the derivative approximation. In the finite difference method, the approximation solutions are found by solving a set of algebraic equations that are the discrete representation of the governing differential equations and the boundary conditions. The discrete representation is formed by replacing the derivatives in the governing equations and the boundary conditions with approximations expressed in terms of difference between nodal displacements [5].

The finite difference method for the nonlinear equation requires the replacement of y'' and y' by difference quotients, which results in a nonlinear system. This system can be solved using successive iterative method and Newton's method.

Newton's method, also known as the Newton–Raphson method, named after Isaac Newton and Joseph Raphson, is a root-finding algorithm which produces successively better approximations to the roots (or zeros) of a real-valued function. It is a powerful technique for solving equations numerically. It is based on the simple idea of linear approximation. The Newton method, properly used on a root with devastating efficiency. The Newton-Raphson method is widely used in finding the root of nonlinear equations. Newton's method converges quadratically. While carrying out this method the system converges quite rapidly once the approximation is close to the actual solution of the nonlinear system. This is seen as a advantage because Newton's method required less iterations, compared to another method with a lower rate of convergence, to reach the solution. However, when the system does not converge then an error in the computations occurs or a solution may not exist [6].

2.1 Advantages and Disadvantages of Finite Difference Method

An important advantage of the finite difference method is its simplicity. Another advantage is the possibility to easily obtain high order approximations, and hence to achieve high order accuracy of the spatial discretization. On the other hand, because the method requires a structured grid, the range of application is clearly restricted. Furthermore, the finite difference method cannot be directly applied in body-fitted(curvilinear) coordinates, but the governing equation have to be first transformed into a Cartesian coordinate system. The problem herewith is that the Jacobian coordinate transformation appears in the flow equations. This Jacobian has to be discretized consistently in order to avoid the introduction of additional numerical errors.

Finite difference methods are the easiest numerical method to understand and implement differential equations, for problems that satisfy its structured discretization assumptions, and can be useful in other domain when we need to estimate other derivatives. It is most transparent and the most general method among the various numerical approaches. It has a straight forward nature and a minimum requirement on hardware.

The problem with finite difference method is that in their most basic form, aren't applicable to unstructured domains. It is difficult to solve large, sparse system of matrices. Approximation property will ensure the error (difference between exact solution and finite difference method). They quickly become unwieldy if we need to start adding any sort of complexity like moving boundaries or an unstructured grid.

The FDM has better stability characteristics, but they generally requires more computation to a specified accuracy. The approximations may not be as accurate as the other numerical method for non-linear equation, there is less sensitivity to round off error.

2.2 Applications

1. Finite difference method's are very viable numerical methods for solution of partial differential equation and hence is suitable for solving plate binding equation. This method is sufficiently accurate for this thin plate analysis.
2. It is used in Power-flow problem formulation. Due to the nonlinear nature of this problem, numerical methods are employed to obtain a solution that is within an acceptable tolerance. The solution to the Power-flow problem begins with identifying the known and unknown variables in the system. Hence FDM's are used.
3. Finite difference methods (FDM) are used to numerically solve the elastodynamic wave equations. Finite difference techniques are applied to approximate both the time and space derivatives and are combined in various ways to provide different numerical algorithms for modeling elastic wave propagation.
4. The finite difference method is directly applicable only to rather simple geometries. Nowadays, it is utilized in the research of turbulent flows and together with immersed boundary cells in biology.
5. The FDM is a time-domain technique, which can find the concentration of dye everywhere in the computational domain at a given time frame. Burley etal.,Wai and Vosoughi solved their convective dye transfer model equation by the FDM presented the results in the form of a number of graphs representing the variation in the concentration of dye at various points in the dyeing machine with time. Shannon etal. used the finite difference method to obtain the solution of their flow model equations, which predicts pressure and velocity profiles based on user defined package geometry, permeability profile and fluid properties. [7]

2.3 Theorem

Suppose the function f in the boundary value problem,

$$y'' = f(x, y, y'),\ a \leq x \leq b,\ y(a) = \alpha \text{ and } y(b) = \beta$$

is continuous on the set,

$$D = ((x, y, y')|a \leq b, -\infty < y < \infty, -\infty < y' < \infty$$

and that the partial differential derivatives f_y and $f_{y'}$ are also continuous in D. If,

1. $f_y(x, y, y') > 0$ for all $(x, y, y') \in D$, and
2. a constant M exists with $|f_y(x, y, y')| < M \forall\ (x, y, y') \in D$, then the boundary value problem has a unique solution.

The numerical method we will be looking at is the finite difference method. This method can be used to solve both linear and nonlinear ordinary differential equations. Here we are considering the nonlinear finite difference method. Let nonlinear boundary value problem is of the form,

$$y'' = f(x, y, y'),\ a \leq x \leq b,\ y(a) = \alpha \text{ and } y(b) = \beta.$$

In order for the finite difference method to be carried out we have to assume f satisfies the following conditions,

1. f and the partial derivatives f_y and $f_{y'}$ are all continuous on

$$D = \{(x, y, y')|a \leq x \leq b, -\infty < y, y' < \infty\}$$

2. $f_y(x, y, y') \geq \delta$ on D, for some $\delta > 0$.
3. Constants k and L exists, with

$$k = \max_{(x,y,y')\in D}|f_y(x, y, y')| \text{ and } L = \max_{(x,y,y')\in D}|f_{y'}(x, y, y')|[8]$$

3 Methodology

Consider the nonlinear boundary value problems (BVPs) for the second order differential equation of the form,

$$y'' = f(x, y, y'),\ a \leq x \leq b, y(a) = \alpha \text{ and } y(b) = \beta \tag{1}$$

Consider the finite-difference method for $y'(x)$ and $y''(x)$ be

$$y'(x) = \frac{1}{2h}(y(x + h - y(x - h)) - \frac{h^2}{6}y'''(x^{\star\star\star}) \tag{2}$$

$$y''(x) = \frac{1}{h^2}(y(x + h) - 2y(x) + y(x - h)) - \frac{h^2}{12}y'''x^{\star\star\star}) \tag{3}$$

where $x^{\star\star\star}$ is between $x - h$ and $x + h$.

By neglecting the higher order terms in equation (2) we get,

Forward difference approximation is,

$$y' = \frac{y(x + h) - y(x)}{h} \tag{4}$$

Backward difference approximation is,

$$y' = \frac{y(x) - y(x + h)}{h} \tag{5}$$

Adding equation (4) and (5) we obtain,

$$y'' = \frac{y(x + h) - y(x - h)}{2h}. \tag{6}$$

Equation (6) is called central difference approximation for first order derivative.

By neglecting the higher order terms in equation (3) we get,

$$y'' = \frac{y(x + h) - 2y(x) + y(x - h)}{h^2} \tag{7}$$

Equation (7) is called central difference approximation for second order derivative.
Similar to the Finite difference method for linear boundary value problem we have, $h = \dfrac{b - a}{N + 1}$ and

$$x_0 = a$$

$$\begin{aligned} x_1 &= a + h \\ &\vdots \\ x_N &= a + Nh \\ x_{N+1} &= a + (N+1)h = b \end{aligned}$$

Let, $y_0 = \alpha$ and $y_{N+1} = \beta$,

Apply central difference approximation formula to the equation (1) we get

$$\frac{1}{h^2}\left(y_{i+1} - 2_i + y_{i-1}\right) = f\left(x_i, y_i, \frac{1}{2h}(y_{i+1} - y_{i-1})\right) \tag{8}$$

$$y_{i-1} - 2y_i + yi + 1 - h^2 f\left(x_i, y_i, \frac{1}{2h}(y_{i+1} - y_{i-1}\right) = 0$$

$$- y_{i-1} + 2y_i - yi + 1 + h^2 f(x_i, y_i, \frac{1}{2h}(y_{i+1} - y_{i-1}) = 0. \tag{9}$$

for $i = 1, 2, \cdots N$.

The $N \times N$ nonlinear system of equations obtained from this method is,

$$\begin{aligned} 1 - y_2 + h^2 f\left(x_1, y_1, \frac{y_2 - \alpha}{2h}\right) - \alpha = 0, \\ -y_1 + 2y_2 - y_3 + h^2 f\left(x_2, y_2, \frac{y_3 - y_1}{2h}\right) = 0, \\ \vdots \\ -y_{N-2} + 2y_{N-1} - y_N = h^2 f\left(x_{N-1}, y_{N-1}, \frac{y_N - y_{N-2}}{2h}\right) = 0, \\ -y_{N-1} + 2y_N + h^2 f\left((x_N, y_N, \frac{\beta - y_{N-1}}{2h}\right) - \beta = 0. \end{aligned} \tag{10}$$

Equation (10) can be written in matrix form as,

$$\text{diag}\left\{-1, 2, -1\right\} Y + h^2 F(x, Y) = AY + h^2 F(x, Y) = \overline{Y}. \tag{11}$$

where,
$$\begin{aligned} Y &= \left[y_1, \cdots y_N\right], \\ \overline{Y} &= \left[\alpha,\ 0 \cdots\ 0,\ \beta\right], \\ A &= \text{diag}\left\{-1, 2, -1\right\} \end{aligned}$$

and

$$F(x, Y) = [F_1(x, Y)\ \cdots\ F_N(x, Y)], \quad \text{where} \quad F_i(x, Y) = f\left(x_i, y_i, \frac{1}{2h}(y_{i+1} - y_{i-1})\right). \tag{12}$$

We can find the initial approximation Y_k by the following equation,

$$Y_k = \alpha + \frac{\beta - \alpha}{b - a}(x_i - \alpha)$$

where, $x_i = \alpha + ih \; \forall \; i = 1, 2, \cdots N$.

It can be shown that the system of N nonlinear equations from (10) has a unique solution if $h < \dfrac{2}{L}$ where,

$$\left|\frac{\partial f(x,y,y')}{\partial y'}\right| \leq L \quad \text{for all } (x,y,y') \text{ in } D = \{(x,y,y') | a \leq x \leq b, -\infty < y, y' < \infty\}$$

Generally, the nonlinear equation in (10) cannot be solved exactly and then $y_1^{(k)} \cdots y_N^{(k)}$ are solved iteratively.

Let the initial values be $Y_0 = \left[y_1^{(0)} \cdots y_N^{(0)}\right]$, we have $Y_k = \left[y_1^{(k)} \cdots y_N^{(k)}\right]$ at the k^{th} iteration and solve Y_{k+1} using following methods.

3.1 Successive iteration method

Y_{k+1} is the solution of the following system of linear equations,

$$AY_{k+1} + h^2 F(x, Y_k) = \overline{Y}$$
$$AY_{k+1} = \overline{Y} - h^2 F(x, Y_k)$$

3.2 Newton method

Let $G(x, Y) = AY + h^2 F(x, Y) - \overline{Y} = [g_1(x, Y) \cdots g_N(x, Y)]$.

The linearization of $G(x, y)$ at Y_k is,

$$G(x, Y) \approx G(x, y_k) + J(x, Y_k)(Y_{k+1} - Y_k).$$

where,

$$J(x, Y) = \left[\frac{\partial g_i(x, Y)}{\partial y_j}\right] = \begin{bmatrix} \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_2} & \cdots & \frac{\partial g_1}{\partial y_N} \\ \frac{\partial g_2}{\partial y_1} & \frac{\partial g_2}{\partial y_2} & \cdots & \frac{\partial g_2}{\partial y_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g_N}{\partial y_1} & \frac{\partial g_N}{\partial y_2} & \cdots & \frac{\partial g_N}{\partial y_N} \end{bmatrix}$$

is the Jacobi Matrix of $G(x, Y)$. Then solve the system of linear equations for Y_{k+1}.

$$G(x, y_k) + J(x, Y_k)(Y_{k+1} - Y_k) = 0$$
$$Y_{k+1} = Y_k - [J(x, Y_k)]^{-1} G(x, Y_k).$$

where,

$$J(x, Y) = A + h^2 J_F(x, Y)$$
$$J_F(x, Y) = \left[\frac{\partial F_i(x, Y)}{\partial y_j}\right]$$
$$J_F(x, Y) = \text{diag}\left\{\frac{\partial F_i(x, Y)}{\partial y_{j-1}}, \frac{\partial F_i(x, Y)}{\partial y_j}, \frac{\partial F_i(x, Y)}{\partial y_{j+1}}\right\}$$

In finite difference method, $J(y_1, y_2, \cdots, y_N)$ is tridiagonal with ij^{th} entry. This means that there are non-zero entries on the diagonal below the main diagonal, and there are non-zero entries on the diagonal directly above the main diagonal. [8]

4 Examples and Discussions

Example 1: Use the nonlinear finite difference method with $h = 0.25$ to approximate the solution to the boundary-value problem,

$$y'' = 2y^3,\ -1 \leq x \leq 0,\ y(-1) = \frac{1}{2},\ y(0) = \frac{1}{3}.$$

Compare the results to the actual solution $y(x) = \dfrac{1}{x+3}$.

Given $y'' = 2y^3,\ -1 \leq x \leq 0,\ y(-1) = \dfrac{1}{2},\ y(0) = \dfrac{1}{3}$

Let $\overline{x} = [-1, -0.75, -0.5, -0.25, 0]$ and $\overline{Y} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{3} \end{bmatrix}$

$$Y_k = \alpha + \frac{\beta - \alpha}{b-a}(x_i - a) = \begin{bmatrix} 0.83333 \\ 0.16666 \\ 0.25 \end{bmatrix}$$

$$F(x,y) = \begin{bmatrix} f\left(x_1, y_1, \left(\frac{1}{2h}(y_2 - \alpha)\right)\right) \\ f\left(x_2, y_2, \left(\frac{1}{2h}(y_3 - y_1)\right)\right) \\ f\left(x_3, y_3, \left(\frac{1}{2h}(\beta - y_3)\right)\right) \end{bmatrix} = 2\begin{bmatrix} y_1^3 \\ y_2^3 \\ y_3^3 \end{bmatrix} \text{ and } F(x, Y_k) = \begin{bmatrix} (\frac{1}{12})^3 \\ (\frac{1}{6})^3 \\ (\frac{1}{4})^3 \end{bmatrix}$$

1. Successive iteration method

$$AY_{k+1} = \overline{Y} - h^2 F(x, Y_k)$$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{3} \end{bmatrix} - \frac{1}{16}\begin{bmatrix} \left(\frac{1}{12}\right)^3 \\ \left(\frac{1}{6}\right)^3 \\ \left(\frac{1}{4}\right)^3 \end{bmatrix}$$

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 0.4519 \\ 0.4158 \\ 0.3741 \end{bmatrix}$$

2. Newton method

$$Y_{k+1} = Y_k - [J(x, Y_k)]^{-1} G(x, Y_k)$$
$$G(x, Y) = AY + h^2 F(x, Y) - \overline{Y}$$

Let, $f(x, y, y') = 2y^3$ hence $F(x,y) = \begin{bmatrix} 2y_1^3 \\ 2y_2^3 \\ 2y_3^3 \end{bmatrix}$ and $J(x, y_k) = \begin{bmatrix} \frac{1}{48} & 0 & 0 \\ 0 & \frac{1}{12} & 0 \\ 0 & 0 & \frac{3}{16} \end{bmatrix}$

$$B = A + h^2 J(x, y_k)$$

$$B = \begin{bmatrix} 2.0026 & -1 & 0 \\ -1 & 2.0104 & -1 \\ 0 & -1 & 2.0234 \end{bmatrix}$$

$$b = Ay_k + h^2 F(x, y_k) - \overline{Y}$$

$$b = \begin{bmatrix} -0.4999 \\ 0.00028 \\ 0.00094 \end{bmatrix}$$

$$Y_{k+1} = Y_k - B^{-1} b$$

$$Y_{k+1} = \begin{bmatrix} 0.4551 \\ 0.4114 \\ 0.3705 \end{bmatrix}$$

Exact Solution		Numerical solution	
		Successive iteration	**Newton method**
1	0.44444	0.4519	0.4551
2	0.4	0.4158	0.3705
3	0.3636	0.3741	0.3705

Example 2: Use the nonlinear finite difference method with $h = 0.25$ to approximate the solution to the boundary-value problem,

$$y'' = -e^{-2y}, 1 \leq x \leq 2, y(1) = 0, y(2) = \ln(2).$$

Compare the results to the actual solution $\ln x$.

Given $y'' = -e^{-2y}, 1 \leq x \leq 2, y(1) = 0, y(2) = \ln(2)$

Let, $\overline{x} = [-1, -0.75, -0.5, -0.25, 0]$ and $\overline{Y} = [0, 0, 0.693147]$

$$Y_k = \alpha + \frac{\beta - \alpha}{b - a}(x_i - a) = \begin{bmatrix} 0.173286 \\ 0.3465735 \\ 0.511986025 \end{bmatrix}$$

$$F(x, Y_k) = \begin{bmatrix} f\left(x_1, y_1, \left(\frac{1}{2h}(y_2 - \alpha)\right)\right) \\ f\left(x_2, y_2, \left(\frac{1}{2h}(y_3 - y_1)\right)\right) \\ f\left(x_3, y_3, \left(\frac{1}{2h}(\beta - y_3)\right)\right) \end{bmatrix} = \begin{bmatrix} e^{-2y_1} \\ e^{-2y_2} \\ e^{-2y_3} \end{bmatrix} = \begin{bmatrix} -0.707107905 \\ -0.50000009 \\ -0.353553486 \end{bmatrix}$$

1. Successive iteration method

$$AY_{k+1} = \overline{Y} - h^2 F(x, Y_k)$$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.693147 \end{bmatrix} - \frac{1}{16} \begin{bmatrix} -0.707107905 \\ -0.50000009 \\ -0.353553486 \end{bmatrix}$$

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 0.2275817 \\ 0.4109692 \\ 0.5631066 \end{bmatrix}$$

2. Newton method

$$Y_{k+1} = Y_k - [J(x, Y_k)]^{-1} G(x, Y_k)$$
$$G(x, Y) = AY + h^2 F(x, Y) - \overline{Y}$$

Let, $f(x, y, y') = 2y^3$.

Hence $F(x, y) = \begin{bmatrix} -0.707107905 \\ -0.50000009 \\ -0.353553486 \end{bmatrix}$ and $J(x, Y_k) = \begin{bmatrix} 1.4142113, 0, 0 \\ 0, 1.999999, 0 \\ 0, 02.828426 \end{bmatrix}$

$$B = A + h^2 J(x, Y_k)$$
$$B = \begin{bmatrix} 1.9116118 & -1 & 0 \\ -1 & 1.87500001 & \\ 0 & -1 & 1.8232234 \end{bmatrix}$$
$$b = Ay_k + h^2 F(x, Y_k) - \overline{Y}$$
$$b = \begin{bmatrix} -0.0441957 \\ -0.0312493 \\ -0.0220971 \end{bmatrix}$$
$$Y_{k+1} = Y_k - B^{-1} b$$
$$Y_{k+1} = \begin{bmatrix} 0.2396983 \\ 0.4293324 \\ 0.5773716 \end{bmatrix}$$

Exact Solution		**Numerical solution**	
		Successive iteration	**Newton method**
1	0.2231435	0.2275817	0.2396983
2	0.4054651	0.4109692	0.4293324
3	0.5596157	0.5631066	0.5773716

Example 3: Use the nonlinear finite difference method with $h = 0.1$ to approximate the solution to the boundary-value problem

$$y'' = 2y^3 - 6y - 2x^3,\ -1 \leq x \leq 2,\ y(1) = 2,\ y(2) = \frac{5}{2} \text{ and } Y_k = [2, 2, 2, 2, 2, 2, 2, 2, 2]$$

Compare the results with the actual solution $x + x^{-1}$.

Given $y'' = 2y^3 - 6y - 2x^3,\ -1 \leq x \leq 2,\ y(1) = 2,\ y(2) = \frac{5}{2},\ Y_k = [2, 2, 2, 2, 2, 2, 2, 2, 2]$

Let, $\overline{x} = [1, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 2]$

$$\overline{Y} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{5}{2} \end{bmatrix}, \quad F(x, Y_k) = \begin{bmatrix} 2y_1^3 & -6y_1 & -2x_1^3 \\ 2y_2^3 & -6y_2 & -2x_2^3 \\ 2y_3^3 & -6y_3 & -2x_3^3 \\ 2y_4^3 & -6y_4 & -2x_4^3 \\ 2y_5^3 & -6y_5 & -2x_5^3 \\ 2y_6^3 & -6y_6 & -2x_6^3 \\ 2y_7^3 & -6y_7 & -2x_7^3 \\ 2y_8^3 & -6y_8 & -2x_8^3 \\ 2y_9^3 & -6y_9 & -2x_9^3 \end{bmatrix}, \quad F(x, Y_0) = \begin{bmatrix} 1.8 \\ 1.6 \\ 1.4 \\ 1.2 \\ 1 \\ 0.8 \\ 0.6 \\ 0.4 \\ 0.2 \end{bmatrix}$$

1. Successive iteration method

$$AY_{k+1} = \overline{Y} - h^2F(x, Y_k)$$

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \\ y_5' \\ y_6' \\ y_7' \\ y_8' \\ y_9' \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{5}{2} \end{bmatrix} - \frac{1}{16} \begin{bmatrix} 1.8 \\ 1.6 \\ 1.4 \\ 1.2 \\ 1 \\ 0.8 \\ 0.6 \\ 0.4 \\ 0.2 \end{bmatrix}$$

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \\ y_5' \\ y_6' \\ y_7' \\ y_8' \\ y_9' \end{bmatrix} = \begin{bmatrix} 1.993 \\ 2.004 \\ 2.031 \\ 2.072 \\ 2.125 \\ 2.188 \\ 2.259 \\ 2.336 \\ 2.417 \end{bmatrix}$$

2. Newton method

$$Y_{k+1} = Y_k - [J(x, Y_k)]^{-1}G(x, Y_k)$$
$$G(x, Y) = AY + h^2F(x, Y) - \overline{Y}$$

Let, $f(x, y, y') = 2y^3 - 6y - 2x^3$

$$F(x, Y_K) = \begin{bmatrix} 2y_1^3 & -6y_1 & -2x_1^3 \\ 2y_2^3 & -6y_2 & -2x_2^3 \\ 2y_3^3 & -6y_3 & -2x_3^3 \\ 2y_4^3 & -6y_4 & -2x_4^3 \\ 2y_5^3 & -6y_5 & -2x_5^3 \\ 2y_6^3 & -6y_6 & -2x_6^3 \\ 2y_7^3 & -6y_7 & -2x_7^3 \\ 2y_8^3 & -6y_8 & -2x_8^3 \\ 2y_9^3 & -6y_9 & -2x_9^3 \end{bmatrix}, \quad J(x, Y_k) = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

$$B = A + h^2J(x, Y_k)$$

$$B = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} + \frac{1}{100} \begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} 2.060 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2.06 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2.060 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2.060 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2.06 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2.06 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2.06 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2.06 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2.06 \end{bmatrix}$$

$$b = Ay_k + h^2F(x, Y_k) - \overline{Y}$$

$$b = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} + \frac{1}{100} \begin{bmatrix} 1.8 \\ 1.6 \\ 1.4 \\ 1.2 \\ 1 \\ 0.8 \\ 0.6 \\ 0.4 \\ 0.2 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{5}{2} \end{bmatrix}$$

$$b = \begin{bmatrix} 0.018 \\ 0.016 \\ 0.014 \\ 0.012 \\ 0.01 \\ 0.008 \\ 0.006 \\ 0.004 \\ -0.498 \end{bmatrix}$$

$$Y_{k+1} = y_k - B^{-1}b$$

$$Y_{k+1} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 2.060 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2.06 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2.060 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2.060 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2.06 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2.06 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2.06 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2.06 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2.06 \end{bmatrix}^{-1} \begin{bmatrix} 0.018 \\ 0.016 \\ 0.014 \\ 0.012 \\ 0.01 \\ 0.008 \\ 0.006 \\ 0.004 \\ -0.498 \end{bmatrix}$$

$$Y_{k+1} = \begin{bmatrix} 1.9814221 \\ 1.9797295 \\ 1.9928207 \\ 2.0194811 \\ 2.0593104 \\ 2.1126983 \\ 2.1808481 \\ 2.2658488 \\ 2.3708004 \end{bmatrix}$$

Exact Solution		**Numerical solution**	
		Successive iteration	**Newton method**
1	2.00	1.993	1.981421
2	2.033	2.004	1.9797295
3	2.069	2.031	1.9928207
4	2.114	2.072	2.0194811
5	2.166	2.125	2.0593104
6	2.225	2.188	2.1126983
7	2.288	2.259	2.1808481
8	2.355	2.336	2.2658488
9	2.246	2.417	2.3708004

5 Inference

In this chapter we studied about finite difference method which is a powerful method in not only solving nonlinear algebraic equations with one variable, but also systems of nonlinear algebraic equations. Finite difference methods are also influential in solving for boundary value problems of nonlinear ordinary differential equations. In finite difference method we can solve the nonlinear system by three methods they are successive iteration method, Newton's method and Crout factorization algorithm. Here, the finite difference method implements both Successive iteration method and Newton's method once the boundary value problem was converted into a nonlinear algebraic system of equations. By the above solved problems we can conclude that successive iteration method provide highly accurate values in less number of iteration as compared with Newton's method. Several numerical examples are solved to illustrate the efficiency and the performance of the finite difference method. It has better stability than shooting methods for boundary value problems. Higher-order differences or extrapolation can be used to improve accuracy. Finite difference method tend to less sensitive to round off error than shooting method.

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Volume 3, Issue 3 **September 2020**